Introduction to Machine Learning
Fall 2025
University of Science and Technology of China

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Notice, to get the full credits, please present your solutions step by step.

Exercise 1: Affine Sets

1. (a) Let $U \subset \mathbb{R}^n$ with $\mathbf{0} \in U$. Show that U is an affine set if and only if U is a (linear) subspace of $\mathbb{R}^n$.

   (b) Let $U \subset \mathbb{R}^n$ be a nonempty affine set. Show that there exists a *unique* subspace $V \subset \mathbb{R}^n$ (called the direction space of U) such that for every $\mathbf{u} \in U$,
   $$U = \mathbf{u} + V.$$
   Equivalently, for any fixed $\mathbf{u}_0 \in U$ one may take $V = U - \mathbf{u}_0$, and this V does not depend on the choice of $\mathbf{u}_0$.

2. (**Linear equations and affine sets**)

   (a) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. The solution set $C = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$ is an affine set.

   (b) Any affine set can be represented as the solution set of a system of linear equations. That is, for any affine set $U \subset \mathbb{R}^n$, there exists $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ such that the solution set $C = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\} = U$, where $m \le n$.
   Hint: just think about how to use some kind of base vectors to represent an affine set.

3. Let $D = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{H}\mathbf{x} = \mathbf{d}\}$ be an affine set in $\mathbb{R}^n$, where $\mathbf{H} \in \mathbb{R}^{m\times n}$ has full row rank m, and $\mathbf{d} \in \mathbb{R}^m$.

   (a) Show that the projection $\mathbf{p} = \mathbf{argmin}_{\mathbf{x}\in D} \|\mathbf{y} - \mathbf{x}\|_2^2$ of a point $\mathbf{y} \in \mathbb{R}^n$ onto D is given by
   $$\mathbf{p} = \mathbf{y} - \mathbf{H}^\top(\mathbf{H}\mathbf{H}^\top)^{-1}(\mathbf{H}\mathbf{y} - \mathbf{d}).$$

   (b) In $\mathbb{R}^2$, let $D = \{\mathbf{x} : x_1 - x_2 = 1\}$ and $\mathbf{y} = (0, 2)$. Compute the projection $\mathbf{p}$ of $\mathbf{y}$ onto D and draw a sketch showing $D, \mathbf{y}, \mathbf{p}$, and the residual vector $\mathbf{y} - \mathbf{p}$.

Solution: ■

Exercise 2: Convex Sets

1. Let $C \subset \mathbb{R}^n$ be a nonempty convex set. Please show the following statements. Some operations that preserve convexity.

   (a) Both $\mathbf{cl}\ C$ and $\mathbf{int}\ C$ are convex.

   (b) The intersection $\bigcap_{i \in I} C_i$ of any collection $\{C_i : i \in \mathcal{I}\}$ of convex sets is convex.

   (c) The set $\{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{a}, \mathbf{x} \in C\}$ is convex, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{a} \in \mathbb{R}^m$.

   (d) The set $\{\mathbf{y} \in \mathbb{R}^m : \mathbf{x} = \mathbf{B}\mathbf{y} + \mathbf{b}, \mathbf{x} \in C\}$ is convex, where $\mathbf{B} \in \mathbb{R}^{n \times m}$ and $\mathbf{b} \in \mathbb{R}^n$.

2. Please find the interior and relative interior of the following convex sets (you don't need to prove them).

   (a) $\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1, x_3 = 0\} \subset \mathbb{R}^3$.

   (b) $\{\mathbf{A} \in S_{++}^n : \mathrm{Tr}(\mathbf{A}) = 1\} \subset \mathbb{R}^{n \times n}$.

   (c) $\{\mathbf{A} \in S_{++}^n : \mathrm{Tr}(\mathbf{A}) = 1\} \subset S^n$.

   (d) (Optional) $\{\mathbf{A} \in S_{++}^n : \mathrm{Tr}(\mathbf{A}) \leq 1\} \subset \mathbb{R}^{n \times n}$.

Solution: ■

Exercise 3: Farkas' Lemma

Let $\mathbf{A} = (\mathbf{a}_1, \cdots, \mathbf{a}_n) \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Consider a set $A = \{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$. Its conic hull $\mathbf{cone}\, A$ is defined as

$$\mathbf{cone}\, A = \{\sum_{i=1}^{n} \alpha_i \mathbf{a}_i : \alpha_i \geq 0, \mathbf{a}_i \in A\}.$$

1. Please show that $\mathbf{cone}\, A$ is closed and convex.

2. If $\mathbf{b} \in \mathbf{cone}\, A$, please show that there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

3. If $\mathbf{b} \notin \mathbf{cone}\, A$, use separation theorems to show that there exists $\mathbf{y} \in \mathbb{R}^m$, such that $\mathbf{A}^\top \mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.

4. Now you can prove Farkas' Lemma: for given $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, one and only one of the two statements hold:

   - $\exists \mathbf{x} \in \mathbb{R}^n$, $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
   - $\exists \mathbf{y} \in \mathbb{R}^m$, $\mathbf{A}^\top \mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.

Solution: ■

Exercise 4: Relative Interior and Closure

1. Let $C \subset \mathbb{R}^n$ be a nonempty convex set.

   (a) Let $\mathbf{x}_0 \in C$. Please show the following statements. The point $\mathbf{x}_0 \in \textbf{relint } C$ if and only if there exists $r > 0$ such that $\mathbf{x}_0 + r\mathbf{v} \in C$ for any $\mathbf{v} \in \textbf{aff}\, C - \mathbf{x}_0$ and $\|\mathbf{v}\|_2 \leq 1$.

   (b) Please show that $\mathbf{x} \in \textbf{relint } C$ if and only if for any $\mathbf{y} \in C$, there exists $\gamma > 0$ such that $\mathbf{x} + \gamma(\mathbf{x} - \mathbf{y}) \in C$.
   Hint: the result in Question (a) may be useful.

   (c) Please show that if $\mathbf{x} \in \textbf{relint } C$, $\mathbf{y} \in \textbf{cl } C$, then $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in \textbf{relint } C$ for $\lambda \in (0, 1]$.
   Hint: there exists $r > 0$, such that $B(\mathbf{x}, r) \cap \textbf{aff } C \subset \textbf{relint } C$. Then consider the convex hull of $(B(\mathbf{x}, r) \cap \textbf{aff } C) \cup \{\mathbf{y}\}$.

2. (Optional) Let $C \subset \mathbb{R}^n$ be a convex set. We already know that $\textbf{cl } C$ and $\textbf{int } C$ are convex. Please show the following statements.

   (a) $\textbf{cl}\,(\textbf{int } C) = \textbf{cl } C$, provided $\textbf{int } C \neq \emptyset$.

   (b) $\textbf{int}\,(\textbf{cl } C) = \textbf{int } C$, provided $\textbf{int } C \neq \emptyset$.

Solution: ■

Exercise 5: Supporting Hyperplane

1. From the lecture, we know that there exsits supporting hyperplanes at the boundary point of a convex set. Please solve the following questions.

   (a) Express the closed convex set $\{\mathbf{x} \in \mathbb{R}^2_+ : x_1 x_2 \geq 1\}$ as an intersection of halfspaces.

   (b) Let $C = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_\infty \leq 1\}$, the ∞-norm unit ball in $\mathbb{R}^n$, and let $\hat{\mathbf{x}}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{\mathbf{x}}$ explicitly. (The ∞-norm of a point $\mathbf{x} \in \mathbb{R}^n$ is defined as $\max_{1\leq i\leq n} |x_i|$.)

2. **The set of separating hyperplanes:** Suppose that C and D are disjoint subsets of $\mathbb{R}^n$ (C and D may **not** be the convex sets). Consider the set of $(\mathbf{a}, b) \in \mathbb{R}^{n+1}$ for which $\mathbf{a}^T\mathbf{x} \leq b$ for all $\mathbf{x} \in C$, and $\mathbf{a}^T\mathbf{x} \geq b$ for all $\mathbf{x} \in D$. Show that this set is a convex cone (if there is no hyperplane that separates C and D, the set becomes $\{(\mathbf{0}, 0)\}$).

3. On the linear space of symmetric $n \times n$ matrices S^n, we can define the standard inner product $\mathrm{tr}(XY) = \sum_{i,j=1}^n X_{ij}Y_{ij}$. Show that we can express the positive semi-definite cone S^n_+ as an intersection of halfspaces. Specifically, for $X, Y \in S^n$,

$$\mathrm{tr}(XY) \geq 0 \text{ for all } X \geq 0 \Leftrightarrow Y \geq 0.$$

Exercise 6: Minkowski Sums

Let $C, D \subset \mathbb{R}^n$ be convex sets. Define their Minkowski sum as

$$C + D = \{x + y : x \in C, y \in D\}.$$

1. For each of the following statements, determine if it is **True** or **False**. If true, provide a proof. If false, provide a counterexample.
   (a) $C + D$ is a convex set.
   (b) (Optional) $\textbf{relint}\,(C + D) = \textbf{relint}\ C + \textbf{relint}\ D$.
   (c) $\textbf{cl}\,(C + D) = \textbf{cl}\ C + \textbf{cl}\ D$.
2. (**Part of Minkowski-Weyl Theorem**) A polyhedron P is the intersection of finitely many halfspaces:

$$P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\} \tag{1}$$

   where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$.
   Show that for any polyhedron $P \subset \mathbb{R}^n$, it can be expressed as the Minkowski sum of a finite set of points and a finite set of directions, i.e., there exist finite sets $V = \{v_1, ..., v_k\}$ and $W = \{w_1, .., w_l\}$ s.t.

$$P = \textbf{conv}\ V + \textbf{cone}\, W = \left\{ \sum_{i=1}^{k} \lambda_i v_i + \sum_{j=1}^{l} \mu_j w_j : \lambda_i \geq 0, \mu_j \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

   Hint: Lift the polyhedron to a cone in $\mathbb{R}^{n+1}$ by introducing a homogenizing variable t, then use the definition of polyhedron (1) in $\mathbb{R}^{n+1}$.

Solution: ■

References