Introduction to Machine Learning
Fall 2024
University of Science and Technology of China

Lecturer: Jie Wang — Homework 3
Posted: Oct. 24, 2024 — Due: Oct. 31, 2024

Notice, to get the full credits, please present your solutions step by step.

Exercise 1: Affine Sets

Affine sets are an important concept in convex optimization theory. Please review the definition of affine sets from our lecture and answer the following questions.

1. A subspace is a special case of an affine set, where the subspace must contain the origin, whereas an affine set does not necessarily have to. A good intuitive example is in $\mathbb{R}^3$, where all lines and planes are affine sets, but only those lines and planes that pass through the origin are subspaces. If we extend the discussion to $\mathbb{R}^n$, we find that the conclusion still holds. Based on this, please show the following statements:

   (a) If $U \subset \mathbb{R}^n$ and $\mathbf{0} \in U$, then U is an affine set if and only if it is a subspace.

   (b) If $U \subset \mathbb{R}^n$ is an affine set, there is a unique subspace $V \subset \mathbb{R}^n$ such that $U = \mathbf{u} + V$ for any $\mathbf{u} \in U$.

2. In optimization problems, many linear constraints are often involved, and many optimization problems are frequently transformed into linear algebra problems. This is related to the properties of affine sets. Please show the following statements:

   (a) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. The solution set $C = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$ is an affine set.

   (b) Any affine set can be represented as the solution set of a system of linear equations. That is, for any affine set $U \subset \mathbb{R}^n$, there exists $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ such that the solution set $C = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\} = U$, where $m \leq n$.
   Hint: just think about how to use some kind of base vectors to represent an affine set.

Solution: ■

Exercise 2: Convex Sets

1. Let $C \subset \mathbb{R}^n$ be a nonempty convex set. Please show the following statements. Some operations that preserve convexity.

   (a) Both **cl** C and **int** C are convex.

   (b) The set **relint** C is convex.

   (c) The intersection $\bigcap_{i \in I} C_i$ of any collection $\{C_i : i \in \mathcal{I}\}$ of convex sets is convex.

   (d) The set $\{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{a}, \mathbf{x} \in C\}$ is convex, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{a} \in \mathbb{R}^m$.

   (e) The set $\{\mathbf{y} \in \mathbb{R}^m : \mathbf{x} = \mathbf{B}\mathbf{y} + \mathbf{b}, \mathbf{x} \in C\}$ is convex, where $\mathbf{B} \in \mathbb{R}^{n \times m}$ and $\mathbf{b} \in \mathbb{R}^n$.

2. Please find the interior and relative interior of the following convex sets (you don't need to prove them).

   (a) $\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1, x_3 = 0\} \subset \mathbb{R}^3$.

   (b) $\{\mathbf{A} \in S_{++}^n : \mathrm{Tr}(\mathbf{A}) = 1\} \subset \mathbb{R}^{n \times n}$.

   (c) $\{\mathbf{A} \in S_{++}^n : \mathrm{Tr}(\mathbf{A}) = 1\} \subset S^n$.

   (d) (Optional) $\{\mathbf{A} \in S_{++}^n : \mathrm{Tr}(\mathbf{A}) \leq 1\} \subset \mathbb{R}^{n \times n}$.

Solution: ■

Exercise 3: Relative Interior and Interior

Let $C \subset \mathbb{R}^n$ be a nonempty convex set.

1. Let $\mathbf{x}_0 \in C$. Please show the following statements. The point $\mathbf{x}_0 \in \textbf{relint}\, C$ if and only if there exists $r > 0$ such that $\mathbf{x}_0 + r\mathbf{v} \in C$ for any $\mathbf{v} \in \textbf{aff}\, C - \mathbf{x}_0$ and $\|\mathbf{v}\|_2 \leq 1$.

2. (a) Please show that $\mathbf{x} \in \textbf{relint}\, C$ if and only if for any $\mathbf{y} \in C$, there exists $\gamma > 0$ such that $\mathbf{x} + \gamma(\mathbf{x} - \mathbf{y}) \in C$.
   Hint: the result in Question 1 may be useful.

   (b) Please show that if $\mathbf{x} \in \textbf{relint}\, C$, $\mathbf{y} \in \textbf{cl}\, C$, then $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in \textbf{relint}\, C$ for $\lambda \in (0, 1]$.
   Hint: there exists $r > 0$, such that $B(\mathbf{x}, r) \cap \textbf{aff}\, C \subset \textbf{relint}\, C$. Then consider the convex hull of $(B(\mathbf{x}, r) \cap \textbf{aff}\, C) \cup \{\mathbf{y}\}$.

3. (Optional) Please show the following statements.

   (a) Suppose $\textbf{int}\, C$ is nonempty, then $\textbf{int}\, C = \textbf{int}\,(\textbf{cl}\, C)$.
   Hint: notice that $\textbf{relint}\, C = \textbf{int}\, C$ if $\textbf{int}\, C$ is nonempty, then apply **Ex** 3.2(b). (in fact, the result still holds when $C = \emptyset$.)

   (b) $\textbf{cl}\,(\textbf{relint}\, C) = \textbf{cl}\, C$.
   Hint: you can use **Ex** 3.2(b).

   (c) $\textbf{relint}\,(\textbf{cl}\, C) = \textbf{relint}\, C$.

Solution: ■

Exercise 4: Supporting Hyperplane

1. From the lecture, we know that there exsits supporting hyperplanes at the boundary point of a convex set. Please solve the following questions.

   (a) Express the closed convex set $\{\mathbf{x} \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1\}$ as an intersection of halfspaces.

   (b) Let $C = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_\infty \leq 1\}$, the ∞-norm unit ball in $\mathbb{R}^n$, and let $\hat{\mathbf{x}}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{\mathbf{x}}$ explicitly. (The ∞-norm of a point $\mathbf{x} \in \mathbb{R}^n$ is defined as $\max_{1 \leq i \leq n} |x_i|$.)

2. **The set of separating hyperplanes:** Suppose that C and D are disjoint subsets of $\mathbb{R}^n$ (C and D may **not** be the convex sets). Consider the set of $(\mathbf{a}, b) \in \mathbb{R}^{n+1}$ for which $\mathbf{a}^T\mathbf{x} \leq b$ for all $\mathbf{x} \in C$, and $\mathbf{a}^T\mathbf{x} \geq b$ for all $\mathbf{x} \in D$. Show that this set is a convex cone (if there is no hyperplane that separates C and D, the set becomes $\{(\mathbf{0}, 0)\}$).

Exercise 5: Farkas' Lemma

Let $\mathbf{A} = (\mathbf{a}_1, \cdots, \mathbf{a}_n) \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Consider a set $A = \{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$. Its conic hall $\mathbf{cone}\, A$ is defined as

$$\mathbf{cone}\, A = \{\sum_{i=1}^{n} \alpha_i \mathbf{a}_i : \alpha_i \geq 0, \mathbf{a}_i \in A\}.$$

1. Please show that $\mathbf{cone}\, A$ is closed and convex.
2. If $\mathbf{b} \in \mathbf{cone}\, A$, please show that there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
3. If $\mathbf{b} \notin \mathbf{cone}\, A$, use separation theorems to show that there exists $\mathbf{y} \in \mathbb{R}^m$, such that $\mathbf{A}^\top \mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.
4. Now you can prove Farkas' Lemma: for given $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, one and only one of the two statements hold:
   - $\exists \mathbf{x} \in \mathbb{R}^n$, $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
   - $\exists \mathbf{y} \in \mathbb{R}^m$, $\mathbf{A}^\top \mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.

Solution: ■

References