Introduction to Machine Learning
Fall 2023
University of Science and Technology of China

Lecturer: Jie Wang — Homework 3
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Notice, to get the full credits, please present your solutions step by step.

Exercise 1: Convex Sets

Let $C \subset \mathbb{R}^n$ be a nonempty convex set. Please show the following statements.

1. Please find the interior and relative interior of the following convex sets (you don't need to prove them).

   (a) $\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1, x_3 = 0\} \subset \mathbb{R}^3$.

   (b) $\{\mathbf{A} \in S_{++}^n : \mathrm{Tr}(\mathbf{A}) = 1\} \subset \mathbb{R}^{n\times n}$.

   (c) $\{\mathbf{A} \in S_{++}^n : \mathrm{Tr}(\mathbf{A}) = 1\} \subset S^n$.

   (d) (Optional) $\{\mathbf{A} \in S_{++}^n : \mathrm{Tr}(\mathbf{A}) \leq 1\} \subset \mathbb{R}^{n\times n}$.

2. Some operations that preserve convexity.

   (a) Both **cl** C and **int** C are convex.

   (b) The set **relint** C is convex.

   (c) The intersection $\bigcap_{i\in I} C_i$ of any collection $\{C_i : i \in \mathcal{I}\}$ of convex sets is convex.

   (d) The set $\{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{a}, \mathbf{x} \in C\}$ is convex, where $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{a} \in \mathbb{R}^m$.

   (e) The set $\{\mathbf{y} \in \mathbb{R}^m : \mathbf{x} = \mathbf{B}\mathbf{y} + \mathbf{b}, \mathbf{x} \in C\}$ is convex, where $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{b} \in \mathbb{R}^n$.

Solution: ■

Exercise 2: Affine Sets

Please show the following statements about affine sets.

1. If $U \subset \mathbb{R}^n$ and $\mathbf{0} \in U$, then U is an affine set if and only if it is a subspace.
2. If $U \subset \mathbb{R}^n$ is an affine set, there is a unique subspace $V \subset \mathbb{R}^n$ such that $U = \mathbf{u} + V$ for any $\mathbf{u} \in U$.
3. Let $U = \mathbf{aff}(\{(1,0,0)^\top, (0,1,0)^\top, (0,0,1)^\top\})$. Given a point $\mathbf{x}_0 \in U$, find two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ such that we can represent any vectors $\mathbf{w} \in U$ in the form of $\mathbf{w} = \mathbf{x}_0 + \alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2$ uniquely.

Solution: ■

Exercise 3: Relative Interior and Interior

Let $C \subset \mathbb{R}^n$ be a nonempty convex set.

1. Let $\mathbf{x}_0 \in C$. Please show the following statements. The point $\mathbf{x}_0 \in \textbf{relint}\, C$ if and only if there exists $r > 0$ such that $\mathbf{x}_0 + r\mathbf{v} \in C$ for any $\mathbf{v} \in \textbf{aff}\, C - \mathbf{x}_0$ and $\|\mathbf{v}\|_2 \leq 1$.

2. (a) Please show that $\mathbf{x} \in \textbf{relint}\, C$ if and only if for any $\mathbf{y} \in C$, there exists $\gamma > 0$ such that $\mathbf{x} + \gamma(\mathbf{x} - \mathbf{y}) \in C$.
   Hint: the result in Question 1 may be useful.

   (b) Please show that if $\mathbf{x} \in \textbf{relint}\, C$, $\mathbf{y} \in \textbf{cl}\, C$, then $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in \textbf{relint}\, C$ for $\lambda \in (0, 1]$.
   Hint: there exists $r > 0$, such that $B(\mathbf{x}, r) \cap \textbf{aff}\, C \subset \textbf{relint}\, C$. Then consider the convex hull of $(B(\mathbf{x}, r) \cap \textbf{aff}\, C) \cup \{\mathbf{y}\}$.

3. (Optional) Please show the following statements.

   (a) Suppose $\textbf{int}\, C$ is nonempty, then $\textbf{int}\, C = \textbf{int}\,(\textbf{cl}\, C)$.
   Hint: notice that $\textbf{relint}\, C = \textbf{int}\, C$ if $\textbf{int}\, C$ is nonempty, then apply **Ex** 3.2(b). (in fact, the result still holds when $C = \emptyset$.)

   (b) $\textbf{cl}\,(\textbf{relint}\, C) = \textbf{cl}\, C$.
   Hint: you can use **Ex** 3.2(b).

   (c) $\textbf{relint}\,(\textbf{cl}\, C) = \textbf{relint}\, C$.

Solution: ■

Exercise 4: Relative Boundary

The relative boundary of a set $S \subset \mathbb{R}^n$ is defined as $\textbf{relbd}\ S = \textbf{cl}\ S \setminus \textbf{relint}\ S$. Please show the following statements **or give counter-examples**.

1. For a set $S \subset \mathbb{R}^n$, $\textbf{relbd}\ S \subset \textbf{bd}\ S$.
2. For a set $S \subset \mathbb{R}^n$, $\textbf{relbd}\ S = \textbf{bd}\ S$.
3. For a set $S \subset \mathbb{R}^n$, $\textbf{relbd}\ S = \textbf{relbd}\ \textbf{cl}\ S$.
4. For a set $S \subset \mathbb{R}^n$ and $\mathbf{x}_0 \in \textbf{cl}\ S$, we can find a sequence $\{\mathbf{x}_k\} \subset \mathbb{R}^n \setminus \textbf{cl}\ S$ such that $\mathbf{x}_k \to \mathbf{x}_0$ as $k \to \infty$.

Solution: ■

Exercise 5: Supporting Hyperplane

1. From the lecture, we know that there exsits supporting hyperplanes at the boundary point of a convex set. Please solve the following questions.

   (a) Express the closed convex set $\{\mathbf{x} \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1\}$ as an intersection of halfspaces.

   (b) Let $C = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_\infty \leq 1\}$, the ∞-norm unit ball in $\mathbb{R}^n$, and let $\hat{\mathbf{x}}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{\mathbf{x}}$ explicitly. (The ∞-norm of a point $\mathbf{x} \in \mathbb{R}^n$ is defined as $\max_{1 \leq i \leq n} |x_i|$.)

2. On the linear space of symmetric $n \times n$ matrices S^n, we can define the standard inner product $\mathrm{tr}(XY) = \sum_{i,j=1}^n X_{ij} Y_{ij}$. From **Ex7** in **HW2**, we know the positive semi-definite cone S^n_+ isn't a polyhedron. However, please show that we can express S^n_+ as an intersection of halfspaces. Specifically, for $X, Y \in S^n$,

$$\mathrm{tr}(XY) \geq 0 \text{ for all } X \geq 0 \Leftrightarrow Y \geq 0.$$

3. **The set of separating hyperplanes:** Suppose that C and D are disjoint subsets of $\mathbb{R}^n$ (C and D may **not** be the convex sets). Consider the set of $(\mathbf{a}, b) \in \mathbb{R}^{n+1}$ for which $\mathbf{a}^T\mathbf{x} \leq b$ for all $\mathbf{x} \in C$, and $\mathbf{a}^T\mathbf{x} \geq b$ for all $\mathbf{x} \in D$. Show that this set is a convex cone (if there is no hyperplane that separates C and D, the set becomes $\{(\mathbf{0}, 0)\}$).

Exercise 6: Farkas' Lemma

Let $\mathbf{A} = (\mathbf{a}_1, \cdots, \mathbf{a}_n) \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Consider a set $A = \{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$. Its conic hall $\textbf{cone}\, A$ is defined as

$$\textbf{cone}\, A = \{\sum_{i=1}^{n} \alpha_i \mathbf{a}_i : \alpha_i \geq 0, \mathbf{a}_i \in A\}.$$

1. Please show that $\textbf{cone}\, A$ is closed and convex.
2. If $\mathbf{b} \in \textbf{cone}\, A$, please show that there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
3. If $\mathbf{b} \notin \textbf{cone}\, A$, use separation theorems to show that there exists $\mathbf{y} \in \mathbb{R}^m$, such that $\mathbf{A}^\top \mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.
4. Now you can prove Farkas' Lemma: for given $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$, one and only one of the two statements hold:
   - $\exists \mathbf{x} \in \mathbb{R}^n$, $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
   - $\exists \mathbf{y} \in \mathbb{R}^m$, $\mathbf{A}^\top \mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.

Solution: ■

References