

Lecture 07. Subdifferential

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1 Introduction

Many popular ML models involve nondifferentiable objective functions, e.g., Lasso introduced as a special case of weighted least squares models. We generalize the concept of gradient for differentiable functions to the so-called subgradient for nondifferentiable convex functions.

2 Subgradients and Subdifferentials

Definition 1. A function $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ ($\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$) is called *proper* if

1. $\exists \mathbf{x} \in \mathbb{R}^n$, such that $f(\mathbf{x}) < \infty$;
2. $f(\mathbf{x}) > -\infty, \forall \mathbf{x} \in \mathbb{R}^n$.

Definition 2. Let $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ be a proper convex function and let $\mathbf{x} \in \text{dom } f$. A vector $\mathbf{g} \in \mathbb{R}^n$ such that

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle, \forall \mathbf{y} \in \mathbb{R}^n \quad (1)$$

is called a *subgradient* of f at \mathbf{x} .

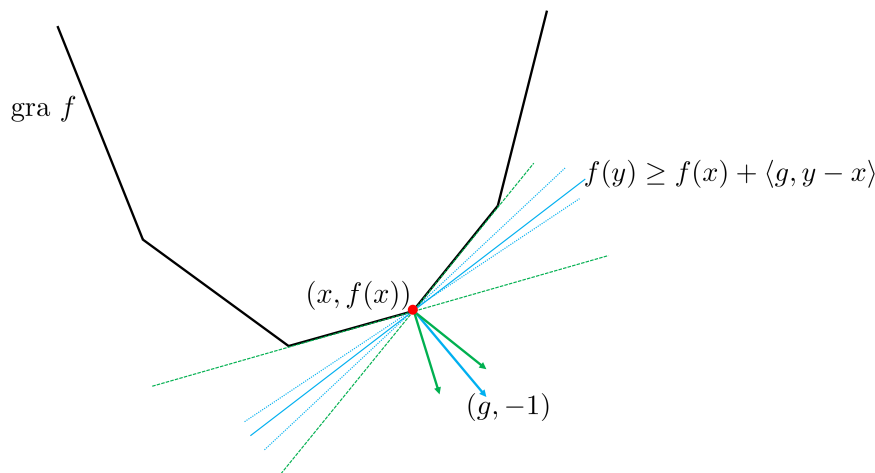


Figure 1: A subgradient.

Question 1. In Definition 2, shall we ask $\mathbf{y} \in \text{dom } f$?

Remark 1. In view of Definition 2, the subgradient is defined for **convex** functions.

Example 1. Consider function $f(x) = |x|, x \in \mathbb{R}$. Find the subgradient of f at 0.

Solution: Let $g \in \partial f(0)$. Then

$$f(y) = |y| \geq f(0) + g(y - 0) = gy.$$

Clearly, the above inequality holds for all $y \in \mathbb{R}$ if and only if $g \in [-1, 1]$. Thus, we have

$$\partial f(0) = [-1, 1],$$

which is not unique. ■

Remark 2 (A geometric interpretation of subdifferential). Inspired by Fig. 1, we can link the subgradient of f to its epigraph. Indeed, for any $(\mathbf{y}, t) \in \text{epi } f$, we have

$$t \geq f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle,$$

which can be rewritten as

$$\left\langle \begin{pmatrix} \mathbf{g} \\ -1 \end{pmatrix}, \begin{pmatrix} \mathbf{y} \\ t \end{pmatrix} - \begin{pmatrix} \mathbf{x} \\ f(\mathbf{x}) \end{pmatrix} \right\rangle \leq 0. \quad (2)$$

The inequality (2) is the **variational inequality** characterizing the projection of a point lying on the ray with base $(\mathbf{x}, f(\mathbf{x}))$ and direction $(\mathbf{g}, -1)$ onto the set $\text{epi } f$.

Furthermore, Fig. 1 implies that the vector $(\mathbf{g}, -1) \in \mathbb{R}^{n+1}$ determines a hyperplane supporting $\text{epi } f$ at the point $(\mathbf{x}, f(\mathbf{x}))$. Can you find the expression of this hyperplane?

Definition 3. The set of all subgradients of f at \mathbf{x} is called the *subdifferential* of f at \mathbf{x} and is denoted by $\partial f(\mathbf{x})$.

Theorem 1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex and $\mathbf{x} \in \text{int}(\text{dom } f)$. Then, f is locally Lipschitz continuous at \mathbf{x} , that is, $\exists \epsilon > 0$ and $M \geq 0$ such that

$$|f(\mathbf{y}) - f(\mathbf{x})| \leq M \|\mathbf{y} - \mathbf{x}\|, \forall \{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| \leq \epsilon\}.$$

Remark 3. The value of the parameter M in Theorem 1 may depend on \mathbf{x} .

Theorem 2. [1] Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex and let $\mathbf{x} \in \text{int}(\text{dom } f)$. Then

1. the subdifferential $\partial f(\mathbf{x})$ is a nonempty, bounded, closed, and convex set;
2. for any $\mathbf{v} \in \mathbb{R}^n$, we have

$$f'(\mathbf{x}; \mathbf{v}) = \lim_{t \downarrow 0} \frac{f(\mathbf{x} + t\mathbf{v}) - f(\mathbf{x})}{t} = \max_{\mathbf{g} \in \partial f(\mathbf{x})} \langle \mathbf{v}, \mathbf{g} \rangle,$$

where $f'(\mathbf{x}; \mathbf{v})$ is the directional derivative of f at \mathbf{x} along the direction \mathbf{v} ;

3. if f is differentiable at \mathbf{x} , then $\partial f(\mathbf{x}) = \{\nabla f(\mathbf{x})\}$.

Proof.

1. We first show that $\partial f(\mathbf{x})$ is **nonempty**. The working horse is the **supporting hyperplane theorem**.

As the point $(\mathbf{x}, f(\mathbf{x}))$ is a boundary point of $\text{epi } f$, the supporting hyperplane theorem implies that we can separate $(\mathbf{x}, f(\mathbf{x}))$ and $\text{epi } f$ by a hyperplane. That is, there exists a $(\mathbf{d}, \alpha) \in \mathbb{R}^{n+1}$ and $(\mathbf{d}, \alpha) \neq 0$ such that

$$\langle (\mathbf{d}, \alpha), (\mathbf{y}, t) \rangle \leq \langle (\mathbf{d}, \alpha), (\mathbf{x}, f(\mathbf{x})) \rangle, \forall (\mathbf{y}, t) \in \text{epi } f,$$

which can be rewritten as

$$\langle \mathbf{d}, \mathbf{y} \rangle + \alpha t \leq \langle \mathbf{d}, \mathbf{x} \rangle + \alpha f(\mathbf{x}), \forall (\mathbf{y}, t) \in \mathbf{epi} f. \quad (3)$$

As the inequality (3) holds for all $(\mathbf{y}, t) \in \mathbf{epi} f$, we conclude $\alpha \leq 0$. We further claim that $\alpha \neq 0$. Suppose not, that is, $\alpha = 0$ (and thus $\mathbf{d} \neq 0$), the inequality (3) becomes

$$\langle \mathbf{d}, \mathbf{y} - \mathbf{x} \rangle \leq 0, \forall (\mathbf{y}, t) \in \mathbf{epi} f. \quad (4)$$

As $\mathbf{x} \in \mathbf{int}(\mathbf{dom} f)$, there exists a small number $\epsilon > 0$ such that $\mathbf{x} + \epsilon \mathbf{d} \in \mathbf{dom} f$. Replacing \mathbf{y} in (4) by $\mathbf{x} + \epsilon \mathbf{d}$ leads to a contradiction. Thus, we must have $\alpha < 0$. Then, by replacing t by $f(\mathbf{y})$ in (3) and dividing both sides by α , we have

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle -\mathbf{d}/\alpha, \mathbf{y} - \mathbf{x} \rangle, \forall \mathbf{y},$$

which implies that $-\mathbf{d}/\alpha \in \partial f(\mathbf{x})$. Therefore, the set $\partial f(\mathbf{x})$ is nonempty.

We next show the **boundedness** of $\partial f(\mathbf{x})$. As $\mathbf{x} \in \mathbf{int}(\mathbf{dom} f)$, we can find a small number $\epsilon_1 > 0$ such that $\{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| < \epsilon_1\} \subseteq \mathbf{dom} f$. Moreover, by Theorem 1, we can find an $\epsilon_2 > 0$ and $M \geq 0$ such that $\forall \|\mathbf{y} - \mathbf{x}\| \leq \epsilon_2$, we have

$$|f(\mathbf{y}) - f(\mathbf{x})| \leq M \|\mathbf{y} - \mathbf{x}\|.$$

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. For any $\mathbf{g} \in \partial f(\mathbf{x})$ and $\mathbf{g} \neq 0$, we choose

$$\mathbf{x}' = \mathbf{x} + \epsilon \mathbf{g} / \|\mathbf{g}\|,$$

which leads to

$$\epsilon \|\mathbf{g}\| = \langle \mathbf{g}, \mathbf{x}' - \mathbf{x} \rangle \leq f(\mathbf{x}') - f(\mathbf{x}) \leq M \|\mathbf{x}' - \mathbf{x}\| = M\epsilon.$$

Thus, $\partial f(\mathbf{x})$ is bounded.

The **closedness** and **convexity** of $\partial f(\mathbf{x})$ can be seen from its definition that, it is the intersection of a set of closed half-spaces.

2. We omit the proof here.
3. For any $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{g} \in \partial f(\mathbf{x})$, we have

$$\langle \nabla f(\mathbf{x}), \mathbf{v} \rangle = f'(\mathbf{x}; \mathbf{v}) \geq \langle \mathbf{g}, \mathbf{v} \rangle.$$

Changing the sign of \mathbf{v} , we conclude that

$$\langle \nabla f(\mathbf{x}), \mathbf{v} \rangle = \langle \mathbf{g}, \mathbf{v} \rangle.$$

By letting $\mathbf{v} = \mathbf{e}_k$, $k = 1, \dots, n$, we have $\mathbf{g} = \nabla f(\mathbf{x})$.

□

Question 2. Consider Theorem 2. The condition that $\mathbf{x} \in \mathbf{int}(\mathbf{dom} f)$ is fundamentally important in deriving the conclusions.

1. If $\mathbf{x} \in \mathbf{dom} f$ but it is not an interior point of $\mathbf{dom} f$, is it possible that $\partial f(\mathbf{x}) = \emptyset$?
2. If $x \in \mathbf{relint}(\mathbf{dom} f)$, is it possible that $\partial f(\mathbf{x})$ is unbounded?



3 Subdifferential Calculus

Lemma 1. [2] Suppose that $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ is a convex function. For $\alpha > 0$, let $h(\mathbf{x}) = \alpha f(\mathbf{x})$. Then, h is convex, and $\partial h(\mathbf{x}) = \alpha \partial f(\mathbf{x})$ for every \mathbf{x} .

Proof. We show the result directly from the definition. Indeed, $\mathbf{g} \in \partial f(\mathbf{x})$ if and only if for all \mathbf{y}

$$h(\mathbf{y}) = \alpha f(\mathbf{y}) \geq \alpha [f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle] = h(\mathbf{x}) + \langle \alpha \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle,$$

which implies that $\alpha \mathbf{g} \in \partial h(\mathbf{x})$. □

Lemma 2. [2] Suppose that $f : \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$ is a convex function, $A \in \mathbb{R}^{m \times n}$, and $\mathbf{b} \in \mathbb{R}^m$. Let $h(\mathbf{x}) = f(A\mathbf{x} + \mathbf{b})$. Then, for any \mathbf{x} , we have

$$\partial h(\mathbf{x}) = A^\top \partial f(A\mathbf{x} + \mathbf{b}).$$

Proof. We show the result directly from the definition. Indeed, we have $\mathbf{g} \in \partial f(A\mathbf{x} + \mathbf{b})$ if and only if

$$h(\mathbf{y}) = f(A\mathbf{y} + \mathbf{b}) \geq f(A\mathbf{x} + \mathbf{b}) + \langle \mathbf{g}, A\mathbf{y} - A\mathbf{x} \rangle = h(\mathbf{x}) + \langle A^\top \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle,$$

which implies that $A^\top \mathbf{g} \in \partial h(\mathbf{x})$. □

Theorem 3 (Moreau-Rockafellar Theorem). [2] Assume that $f = f_1 + f_2$, where $f_i : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$, $i = 1, 2$, are convex proper functions. If there exists a point $\mathbf{x}_0 \in \mathbf{dom} f$ such that f_1 is continuous at \mathbf{x}_0 , then

$$\partial f(\mathbf{x}) = \partial f_1(\mathbf{x}) + \partial f_2(\mathbf{x}), \forall \mathbf{x} \in \mathbf{dom} f.$$

Definition 4. A convex function is called closed if its epigraph is a closed set.

Lemma 3. [1] Let functions $f_i(\mathbf{x})$, $i = 1, \dots, m$, be closed and convex. Then function

$$f(\mathbf{x}) = \max_{1 \leq i \leq m} f_i(\mathbf{x})$$

is also closed and convex. For any $\mathbf{x} \in \mathbf{int}(\mathbf{dom} f) = \bigcap_{i=1}^m \mathbf{int}(\mathbf{dom} f_i)$, we have

$$\partial f(\mathbf{x}) = \mathbf{conv} \{ \partial f_i(\mathbf{x}) : i \in \mathcal{I}(\mathbf{x}) \},$$

where $\mathcal{I}(\mathbf{x}) = \{ i : f_i(\mathbf{x}) = f(\mathbf{x}) \}$.

Lemma 4. Let Δ be a set and

$$f(\mathbf{x}) = \sup \{ \phi(\mathbf{y}, \mathbf{x}) : \mathbf{y} \in \Delta \}.$$

Suppose that for any fixed $\mathbf{y} \in \Delta$, the function $\phi(\mathbf{y}, \mathbf{x})$ is closed and convex in \mathbf{x} . Then, $f(\mathbf{x})$ is closed and convex. For and \mathbf{x} from

$$\mathbf{dom} f = \{ \mathbf{x} \in \mathbb{R}^n : \exists \gamma \text{ such that } \phi(\mathbf{y}, \mathbf{x}) \leq \gamma, \forall \mathbf{y} \in \Delta \},$$

we have

$$\partial f(\mathbf{x}) \supseteq \mathbf{conv} \{ \partial \phi_{\mathbf{x}}(\mathbf{y}, \mathbf{x}) : \mathbf{y} \in \mathcal{I}(\mathbf{x}) \},$$

where $\mathcal{I}(\mathbf{x}) = \{ \mathbf{y} : \phi(\mathbf{y}, \mathbf{x}) = f(\mathbf{x}) \}$. When Δ is compact and $\phi(\mathbf{y}, \mathbf{x}')$ is continuous (upper semi-continuous) in \mathbf{y} for all \mathbf{x}' in a neighborhood of \mathbf{x} , we get an equality above.

Example 2. Consider function $f(x) = |x|$, $x \in \mathbb{R}$. Find $\partial f(x)$.

Solution: Clearly, $f(x)$ is a convex function. We find $\partial f(x)$ by two different approaches.

1. We have derived that $\partial f(0) = [-1, 1]$. Moreover, by noting that $f(x)$ is differentiable for $x \neq 0$, we have

$$\partial f(x) = \begin{cases} 1, & \text{if } x > 0, \\ [-1, 1], & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

2. Let $f_1(x) = x$ and $f_2(x) = -x$. Clearly, we have $\partial f_1(x) = \{\nabla f_1(x)\} = \{1\}$, and similarly $\partial f_2(x) = \{-1\}$.

Moreover, it is easy to see that $f(x) = \max\{f_1(x), f_2(x)\}$, and thus

$$\begin{aligned} \partial f(x) &= \mathbf{conv} \{\partial f_i(x) : f_i(x) = f(x)\} \\ &= \begin{cases} 1, & \text{if } x > 0, \\ [-1, 1], & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases} \end{aligned}$$

■

Example 3. Let $f(\mathbf{x}) = \|\mathbf{x}\|_1$, where $\mathbf{x} \in \mathbb{R}^n$. Find $\partial f(\mathbf{x})$.

Solution: It is easy to see that $f(\mathbf{x})$ is a convex function. We compute $\partial f(\mathbf{x})$ by two different approaches.

1. By Lemma 2 and Theorem 3, we have

$$\begin{aligned} f(\mathbf{x}) &= \|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i| = \sum_{i=1}^n |\mathbf{e}_i^\top \mathbf{x}| \\ \Rightarrow \partial f(\mathbf{x}) &= \partial \left(\sum_{i=1}^n |\mathbf{e}_i^\top \mathbf{x}| \right) = \sum_{i=1}^n \partial |\mathbf{e}_i^\top \mathbf{x}| = \sum_{i=1}^n \mathbf{e}_i \partial |x_i| \\ &= \left\{ \mathbf{v} \in \mathbb{R}^n : v_i = \begin{cases} 1, & \text{if } x_i > 0, \\ [-1, 1], & \text{if } x_i = 0, \\ -1, & \text{if } x_i < 0. \end{cases} \right\} \end{aligned}$$

2. By Lemma 3, we have

$$\begin{aligned} f(\mathbf{x}) &= \|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i| = \max \{ \langle \mathbf{s}, \mathbf{x} \rangle : \mathbf{s} \in \mathbb{R}^n, |s_i| = 1, \forall i \} \\ \Rightarrow \partial f(\mathbf{x}) &= \mathbf{conv} \{ \mathbf{s} : \mathbf{s} \in \mathbb{R}^n, |s_i| = 1, \forall i, \langle \mathbf{s}, \mathbf{x} \rangle = \|\mathbf{x}\|_1 \} \\ &= \left\{ \mathbf{v} \in \mathbb{R}^n : v_i = \begin{cases} 1, & \text{if } x_i > 0, \\ [-1, 1], & \text{if } x_i = 0, \\ -1, & \text{if } x_i < 0. \end{cases} \right\} \end{aligned}$$



Example 4. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by $f(\mathbf{x}) = \max\{x_i, i = 1, \dots, n\}$, where x_i is the i^{th} component of \mathbf{x} .

Solution: To see that $f(\mathbf{x})$ is convex, it suffices to note that

$$f(\mathbf{x}) = \max_{i=1, \dots, n} \langle \mathbf{e}_i, \mathbf{x} \rangle.$$

Thus, by Lemma 3, we have

$$\partial f(\mathbf{x}) = \mathbf{conv} \{ \mathbf{e}_i : \langle \mathbf{e}_i, \mathbf{x} \rangle = f(\mathbf{x}) \} = \{ \mathbf{v} : \mathbf{v} \in \mathbb{R}_+^n, \|\mathbf{v}\|_1 = 1, \langle \mathbf{v}, \mathbf{x} \rangle = f(\mathbf{x}) \}.$$

Example 5. [3] Let $f : \mathbb{S}^n \rightarrow \mathbb{R}$ be defined by $f(X) = \lambda_{\max}(X)$. Find $\partial f(X)$.

Solution: From last lecture, we have shown that $f(X)$ is a convex function. By eigen-decomposition, a symmetric matrix can be written as

$$X = U\Lambda U^\top,$$

where $U^\top U = I$ and $\Lambda = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n$. Let $U = (\mathbf{u}_1, \dots, \mathbf{u}_n)$, i.e., \mathbf{u}_i is the eigenvector corresponding to λ_i . We then write $f(X)$ as the maximum of a set of linear functions over X :

$$\begin{aligned} f(X) &= \max \{ \langle \mathbf{s}, X\mathbf{s} \rangle : \|\mathbf{s}\| = 1 \} \\ &= \max \{ \langle \mathbf{s}\mathbf{s}^\top, X \rangle : \|\mathbf{s}\| = 1 \} \end{aligned}$$

Assume that $\lambda_{\max} = \lambda_1 = \dots = \lambda_r$, where $1 \leq r \leq n$. We can see that

$$\mathbf{u}_i \in \underset{\|\mathbf{s}\|=1}{\mathbf{argmax}} \langle \mathbf{s}\mathbf{s}^\top, X \rangle, \quad i = 1, \dots, r.$$

Let $U^r = (\mathbf{u}_1, \dots, \mathbf{u}_r)$. Then,

$$S^* := \underset{\|\mathbf{s}\|=1}{\mathbf{argmax}} \langle \mathbf{s}\mathbf{s}^\top, X \rangle = \{ \mathbf{v} : \mathbf{v} \in \mathbf{span} U^r, \|\mathbf{v}\| = 1 \} = \{ \mathbf{v} : \mathbf{v} = U^r \mathbf{q}, \mathbf{q} \in \mathbb{R}^r, \|\mathbf{q}\| = 1 \}.$$

By Lemma 4, we have

$$\begin{aligned} \partial f(X) &= \mathbf{conv} \{ \mathbf{v}\mathbf{v}^\top : \mathbf{v} \in S^* \} \\ &= \mathbf{conv} \{ U^r \mathbf{q}\mathbf{q}^\top (U^r)^\top : \mathbf{q} \in \mathbb{R}^r, \|\mathbf{q}\| = 1 \} \\ &= \{ U^r G (U^r)^\top : G \succeq 0, \text{trace } G = 1 \}. \end{aligned}$$



References

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- [2] A. Ruszczyński. *Nonlinear Optimization*. Princeton University Press, 2006.
- [3] G. A. Watson. Characterization of the subdifferential of some matrix norms. *Linear Algebra and Its Applications*, 1992.