

Introduction to Machine Learning
Spring 2021
University of Science and Technology of China

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Homework 2
Due: Apr. 16, 2021

Notice, to get the full credits, please present your solutions step by step.

Exercise 1: Limit and Limit Points

1. Show that $\{\mathbf{x}_n\}$ in \mathbb{R}^n converges to $\mathbf{x} \in \mathbb{R}^n$ if and only if $\{\mathbf{x}_n\}$ is bounded and has a unique limit point \mathbf{x} .
2. (**Limit Points of a Set**). Let C be a subset of \mathbb{R}^n . A point $\mathbf{x} \in \mathbb{R}^n$ is called a limit point of C if there is a sequence $\{\mathbf{x}_n\}$ in C such that $\mathbf{x}_n \rightarrow \mathbf{x}$ and $\mathbf{x}_n \neq \mathbf{x}$ for all positive integers n . If $\mathbf{x} \in C$ and \mathbf{x} is not a limit point of C , then \mathbf{x} is called an isolated point of C . Let C' be the set of limit points of the set C . Please show the following statements.
 - (a) If $C = (0, 1) \cup \{2\} \subset \mathbb{R}$, then $C' = [0, 1]$ and $x = 2$ is an isolated point of C .
 - (b) The set C' is closed.
 - (c) The closure of C is the union of C' and C ; that is $\mathbf{cl} C = C' \cup C$. Moreover, $C' \subset C$ if and only if C is closed.

Solution:



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Exercise 2: Open and Closed Sets

The norm ball $\{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\|_2 < r, \mathbf{x} \in \mathbb{R}^n\}$ is denoted by $B_r(\mathbf{x})$.

1. Given a set $C \subset \mathbb{R}^n$, please show the following are equivalent.
 - (a) The set C is closed; that is $\text{cl } C = C$.
 - (b) The complement of C is open.
 - (c) If $B_\epsilon(\mathbf{x}) \cap C \neq \emptyset$ for every $\epsilon > 0$, then $\mathbf{x} \in C$.
2. Given $A \subset \mathbb{R}^n$, a set $C \subset A$ is called open in A if

$$C = \{\mathbf{x} \in C : B_\epsilon(\mathbf{x}) \cap A \subset C \text{ for some } \epsilon > 0\}.$$

A set C is said to be closed in A if $A \setminus C$ is open in A .

- (a) Let $B = [0, 1] \cup \{2\}$. Please show that $[0, 1]$ is not an open set in \mathbb{R} , while it is both open and closed in B .
- (b) Please show that a set $C \subset A$ is open in A if and only if $C = A \cap U$, where U is open in \mathbb{R}^n .

Solution:



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Exercise 3: Bolzano-Weierstrass Theorem

The Least Upper Bound Axiom

Any nonempty set of real numbers with an upper bound has a least upper bound. That is, $\sup C$ always exists for a nonempty bounded above set $C \subset \mathbb{R}$.

Please show the following statements from **the least upper bound axiom**.

1. Let C be a nonempty subset of \mathbb{R} that is bounded above. Prove that $u = \sup C$ if and only if u is an upper bound of C and

$$\forall \epsilon > 0, \exists a \in C \text{ such that } a > u - \epsilon.$$

2. Every bounded sequence in \mathbb{R} has at least one limit point.
3. Every bounded sequence in \mathbb{R}^n has at least one limit point.

Solution:



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Exercise 4: Extreme Value Theorem

1. Show that a set $C \subset \mathbb{R}^n$ is compact if and only if C is closed and bounded.
2. Let C be a compact subset of \mathbb{R}^n and $f : C \rightarrow \mathbb{R}$ be continuous. Please show that there exist $\mathbf{a}, \mathbf{b} \in C$ such that

$$f(\mathbf{a}) \leq f(\mathbf{x}) \leq f(\mathbf{b}), \forall \mathbf{x} \in C.$$

3. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Show that the range of f is a compact interval $[c, d]$ for some $c, d \in \mathbb{R}$.

Solution:



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Exercise 5: Convex Sets

Let $C \subset \mathbb{R}^n$ be a convex set. Please show the following statements.

1. The intersection $\bigcap_{i \in I} C_i$ of any collection $\{C_i : i \in I\}$ of convex sets is convex.
2. Both $\mathbf{cl} C$ and $\mathbf{int} C$ are convex.
3. The set $\{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{a}, \mathbf{x} \in C\}$ is convex, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{a} \in \mathbb{R}^m$.
4. The set $\{\mathbf{y} \in \mathbb{R}^m : \mathbf{x} = \mathbf{B}\mathbf{y} + \mathbf{b}, \mathbf{x} \in C\}$ is convex, where $\mathbf{B} \in \mathbb{R}^{n \times m}$ and $\mathbf{b} \in \mathbb{R}^n$.

Solution:



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Exercise 6: Continuity of Convex Functions

1. Let f be a convex function defined on an open interval Ω of \mathbb{R} . Show that for any $\alpha < \beta < \gamma$ in Ω , we have

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} \leq \frac{f(\gamma) - f(\alpha)}{\gamma - \alpha} \leq \frac{f(\gamma) - f(\beta)}{\gamma - \beta}$$

2. Let f be a convex function defined on an open interval Ω of \mathbb{R} . Show that f is continuous on Ω .
3. Let f be a convex function defined on an convex open set Ω of \mathbb{R}^n . Show that f is continuous on Ω . [1]

Solution:



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Exercise 7: Strictly Convex Functions

1. Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable. Please show that f is strictly convex if and only if

$$f(\mathbf{y}) > f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{y}.$$

2. Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is twice continuously differentiable. Please show that f is strictly convex if

$$\nabla^2 f(\mathbf{x}) \succ 0, \forall \mathbf{x} \in \mathbb{R}^n.$$

Is the converse true? Please show your statement.

Solution:



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Exercise 8: Strongly Convex Functions

1. Suppose that f is continuously differentiable. Show that a continuously differentiable function f is strongly convex with parameter $\mu > 0$ if and only if

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\mu}{2} \|\mathbf{y} - \mathbf{x}\|_2^2, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

2. Suppose that f is twice continuously differentiable and strongly convex with parameter $\mu > 0$. Please give an interpretation of μ in terms of the eigenvalues of $\nabla^2 f(\mathbf{x})$.
3. (**Lipschitz Continuity**). Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is twice continuously differentiable, and the gradient of f is Lipschitz continuous, i.e.,

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_2 \leq L \|\mathbf{x} - \mathbf{y}\|_2, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

where $L > 0$ is the Lipschitz constant. Please give an interpretation of L in terms of the eigenvalues of $\nabla^2 f(\mathbf{x})$.

Solution:

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Exercise 9: Epigraph

1. Given a real-valued function $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$. Show that the following are equivalent:
 - (a) The epigraph $\mathbf{epi} f$ is closed.
 - (b) The α -level set C_α is closed for any value of α .
2. Let the function $f : \mathbb{R}^n \times \mathbb{S}_{++}^n \rightarrow \mathbb{R}$ be defined as

$$f(\mathbf{x}, \mathbf{Y}) = \mathbf{x}^\top \mathbf{Y}^{-1} \mathbf{x},$$

where \mathbb{S}_{++}^n denotes the set of symmetric positive-definite $n \times n$ matrices. Please show that f is convex.

Solution:



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Exercise 10: Operations that Preserve Convexity

1. Let $f : \mathbb{R}^m \rightarrow (-\infty, +\infty]$ be a given convex function, let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, and let

$$F(\mathbf{x}) = f(\mathbf{Ax} + \mathbf{b}), \mathbf{x} \in \mathbb{R}^n$$

Show that F is convex.

2. Let $f_i : \mathbb{R}^n \rightarrow (-\infty, +\infty]$, $i = 1, \dots, m$ be given convex functions. Show that

$$F(\mathbf{x}) = \sum_{i=1}^m w_i f_i(\mathbf{x})$$

is convex, where $w_i \geq 0$, $i = 1, \dots, m$.

3. (a) Let $f_i : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be given convex functions for $i \in I$, where I is an arbitrary index set. Show that

$$F(\mathbf{x}) = \sup_{i \in I} f_i(\mathbf{x})$$

is convex.

- (b) Consider the function $f(\mathbf{X}) = \lambda_{\max}(\mathbf{X})$, with $\text{dom } f = \mathbb{S}^n$, where $\lambda_{\max}(\mathbf{X})$ is the largest eigenvalue of \mathbf{X} and \mathbb{S}^n is a set of $n \times n$ real symmetric matrices. Please show that f is a convex function.

4. Suppose that the training set is $\{(\mathbf{x}_i, y_i)\}_{i=1}^m$, where $\mathbf{x}_i \in \mathbb{R}^d$ is the i^{th} data instance and $y_i \in \mathbb{R}$ is the corresponding label.

- (a) Lasso refers to the regression problem as follows:

$$\min_{\boldsymbol{\beta}} \frac{1}{2m} \|\mathbf{X}\boldsymbol{\beta} - \mathbf{y}\|_2^2 + \lambda \|\boldsymbol{\beta}\|_1,$$

where $\mathbf{X} \in \mathbb{R}^{m \times d}$ with its i^{th} row being \mathbf{x}_i^\top , $\boldsymbol{\beta} \in \mathbb{R}^d$, and $\lambda > 0$ is the regularization parameter.

- (b) Logistic regression refers to the problem as follows:

$$\min_{\mathbf{w}, b} \frac{1}{m} \sum_{i=1}^m \log(1 + \exp(-y_i(\langle \mathbf{w}, \mathbf{x}_i \rangle - b))),$$

where $y_i \in \{1, -1\}$, $\mathbf{w} \in \mathbb{R}^d$, and $b \in \mathbb{R}$.

Show that the objective functions in the above problems are convex.

Solution:

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References

- [1] Every convex function is locally lipschitz. *The American Mathematical Monthly*, 79(10):1121–1124, 1972.