Introduction to Machine Learning

Fall 2021

University of Science and Technology of China

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Notice, to get the full credits, please present your solutions step by step.

Exercise 1: Convex Sets

Let $C \subset \mathbb{R}^n$ be a convex set. Please show the following statements.

1. Both **cl** C and **int** C are convex.
2. The set **relint** C is convex.
3. The intersection $\bigcap_{i \in I} C_i$ of any collection $\{C_i : i \in \mathcal{I}\}$ of convex sets is convex.
4. If C_1 and C_2 are convex sets in $\mathbb{R}^n$, then the set

$$C_1 - C_2 = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 \in C_1, \mathbf{x}_2 \in C_2\}$$

   is convex.
5. The set $\{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{a}, \mathbf{x} \in C\}$ is convex, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{a} \in \mathbb{R}^m$.
6. The set $\{\mathbf{y} \in \mathbb{R}^m : \mathbf{x} = \mathbf{B}\mathbf{y} + \mathbf{b}, \mathbf{x} \in C\}$ is convex, where $\mathbf{B} \in \mathbb{R}^{n \times m}$ and $\mathbf{b} \in \mathbb{R}^n$.

Solution: ■

Exercise 2: Affine Sets

Please show the following statements about affine sets.

1. If $U \subset \mathbb{R}^n$ and $\mathbf{0} \in U$, then U is an affine set if and only if it is a subspace.
2. If $U \subset \mathbb{R}^n$ is an affine set, there is a unique subspace $V \subset \mathbb{R}^n$ such that $U = \mathbf{u} + V$ for any $\mathbf{u} \in U$.

Solution: ■

Exercise 3: Relative Interior

Let $C \subset \mathbb{R}^n$ be a convex set and $\mathbf{x}_0 \in C$. Please show the following statements.

1. The set $\mathbf{aff}\, C - \mathbf{x}_0$ is a subspace of $\mathbb{R}^n$.
2. The point $\mathbf{x}_0 \in \mathbf{relint}\ C$ if and only if there exists $r > 0$ such that $\mathbf{x}_0 + r\mathbf{v} \in C$ for any $\mathbf{v} \in \mathbf{aff}\, C - \mathbf{x}_0$ and $\|\mathbf{v}\|_2 \leq 1$.
3. Let $\{\mathbf{v}_i\}_{i=1}^m$ be a basis of $\mathbf{aff}\, C - \mathbf{x}_0$. Then $\mathbf{x}_0 \in \mathbf{relint}\ C$ if and only if there exists $r > 0$ such that $\mathbf{x}_0 + r\sum_i \alpha_i \mathbf{v}_i \in C$ for any $\{\alpha_i\}_{i=1}^m$ with $\sum_i \alpha_i^2 \leq 1$.

Solution: ■

Exercise 4: Relative Boundary

The relative boundary of a set $S \subset \mathbb{R}^n$ is defined as $\textbf{relbd } S = \textbf{cl } S \setminus \textbf{relint } S$. Please show the following statements **or give counter-examples**.

1. For a set $S \subset \mathbb{R}^n$, $\textbf{relbd } S \subset \textbf{bd } S$.
2. For a set $S \subset \mathbb{R}^n$, $\textbf{relbd } S = \textbf{bd } S$.
3. For a set $S \subset \mathbb{R}^n$, $\textbf{relbd } S = \textbf{relbd cl } S$.
4. For a convex set $C \subset \mathbb{R}^n$, $\textbf{relbd } C = \textbf{relbd cl } C$.
5. For a set $S \subset \mathbb{R}^n$ and $\mathbf{x}_0 \in \textbf{cl } S$, we can find a sequence $\{\mathbf{x}_k\} \subset \mathbb{R}^n \setminus \textbf{cl } S$ such that $\mathbf{x}_k \to \mathbf{x}_0$ as $k \to \infty$.
6. For a convex set $C \subset \mathbb{R}^n$ and $\mathbf{x}_0 \in \textbf{bd } C$, we can find a sequence $\{\mathbf{x}_k\} \subset \mathbb{R}^n \setminus \textbf{cl } C$ such that $\mathbf{x}_k \to \mathbf{x}_0$ as $k \to \infty$.

Solution: ■

Exercise 5: Minkowski Summation of Sets (Optional)

The Minkowski sum of two sets S_1 and S_2 is defined by

$$S_1 + S_2 = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in S_1, \mathbf{y} \in S_2\}.$$

1. Let $S_1 = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 \leq 1\}$ and $S_2 = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_\infty \leq 1\}$.
   (a) Please draw the set $S_1 + S_2$.
   (b) How do you tell if a point $\mathbf{x}$ is in the set $S_1 + S_2$?
2. Recall that $\mathbb{R}^n$ can be decomposed as $\mathbb{R}^n = S \oplus S^\perp$, i.e., $\mathbb{R}^n = S + S^\perp$ and $S \cap S^\perp = \emptyset$, where $S \subset \mathbb{R}^n$ is a subspace and $S^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \perp S\}$. Let $C \subset \mathbb{R}^n$ be a convex set. Define $\hat{C} = C + (\mathbf{aff}\, C - \mathbf{x}_0)^\perp$. Please show that:
   (a) $\dim(\mathbf{aff}\, \hat{C}) = n$;
   (b) $\mathbf{relint}\ C = \mathbf{relint}\ \hat{C}$;
   (c) $\mathbf{relbd}\ C = \mathbf{relbd}\ \hat{C}$.

Solution: ■

Exercise 6: Convex Sets and Linear Functions

Let $C \subset \mathbb{R}^n$ be a convex set and $l(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle$ be a linear function on $\mathbb{R}^n$. The linear function is nontrivial if $\mathbf{a} \neq \mathbf{0}$. Suppose $\mathbf{x}_0 \in C$ and denote

$$\mathcal{B}_C(\mathbf{x}_0, r) = \mathcal{B}(\mathbf{x}_0, r) \cap \mathbf{aff}\, C.$$

Please show the following statements.

1. If $l(\mathbf{x}) = \alpha, \forall\, \mathbf{x} \in \mathcal{B}_C(\mathbf{x}_0, r)$, then $l(\mathbf{x}) = \alpha, \forall\, \mathbf{x} \in C$.
2. The linear function $l(\mathbf{x}) = \alpha, \forall\, \mathbf{x} \in \mathcal{B}_C(\mathbf{x}_0, r)$ for some constant α if and only if $\mathbf{a} \perp (\mathbf{aff}\, C - \mathbf{x}_0)$.
3. The linear function $l(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle$ is not constant if and only if $\Pi_{(\mathbf{aff}\, C - \mathbf{x}_0)}(\mathbf{a}) \neq \mathbf{0}$, where Π means the projection.
4. If $\mathbf{relbd}\, C \neq \emptyset$, then there exists a nontrivial linear function l, and a constant α such that $l(\mathbf{x}) \leq \alpha$ for $\forall\, \mathbf{x} \in C$.

Solution: ■

Exercise 7: Convex Set and Unit Ball (Optimal)

Let $C \subset \mathbb{R}^n$ be a nonempty **compact** convex set. Without loss of generality, we can assume that $\mathbf{0} \in C$.

1. Please show that C must have a relative interior. (**Hint:** Take $\{\mathbf{e}_i\}_{i=1}^m \subset C$ such that it is a basis of $\mathbf{aff}\, C$. Then consider $\mathbf{e}_0 = \frac{1}{m+1} \sum_{i=1}^m \mathbf{e}_i$.)

2. Without loss of generality, suppose $\mathbf{0}$ is a relative interior of C. Then the Minkowski functional $p : \mathbf{aff}\, C \to \mathbb{R}_{\geq 0}$ is defined as $p(\mathbf{x}) = \inf\{\lambda > 0 : \frac{\mathbf{x}}{\lambda} \in C\}$. Please prove its following properties:

   (a) $p(\mathbf{x}) = 0 \iff \mathbf{x} = \mathbf{0}$.

   (b) p is continuous on $\mathbf{aff}\, C$.

   (c) There exist constants c_1 and c_2 such that $c_1 \|\mathbf{x}\|_2 \leq p(\mathbf{x}) \leq c_2 \|\mathbf{x}\|_2$ for $\forall \mathbf{x} \in \mathbf{aff}\, C$.

3. Still we suppose that $\mathbf{0}$ is a relative interior of C. Let $\overline{\mathcal{B}}_1(0) = \{\mathbf{x} \in \mathbf{aff}\, C : \|\mathbf{x}\|_2 \leq 1\}$. Define $\varphi : \overline{\mathcal{B}}_1(0) \to C$ as

$$\varphi(\mathbf{z}) = \begin{cases} \frac{\|\mathbf{z}\|_2}{p(\mathbf{z})}\mathbf{z}, & \mathbf{z} \neq \mathbf{0} \\ 0, & \mathbf{z} = \mathbf{0}. \end{cases}$$

   Please show that φ is a homeomorphism, i.e., φ is a bijection and both φ and φ^{-1} are continuous.

Solution: ■

Exercise 8: Separation Theorems

Let $C_1, C_2, C \subset \mathbb{R}^n$ be convex sets. Please show the following statements.

1. If C_1 is compact and $C_1 \cap C_2 \neq \emptyset$, then C_1 and C_2 can be strongly separated.
2. The sets C_1 and C_2 can be properly separated if and only if $\textbf{relint } C_1 \cap \textbf{relint } C_2 = \emptyset$.
3. If $\dim(\textbf{aff}\, C) = n$ and $\mathbf{x} \in \mathbb{R}^n \setminus C$, then $\mathbf{x}$ and C can be properly separated.

Solution: ■

Exercise 9: Farkas' Lemma

Let $\mathbf{A} = (\mathbf{a}_1, \cdots, \mathbf{a}_n) \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Consider a set $A = \{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$. Its conic hall $\mathbf{cone}\, A$ is defined as

$$\mathbf{cone}\, A = \{\sum_{i=1}^{n} \alpha_i \mathbf{a}_i : \alpha_i \geq 0, \mathbf{a}_i \in A\}.$$

1. Please show that $\mathbf{cone}\, A$ is closed and convex.
2. If $\mathbf{b} \in \mathbf{cone}\, A$, please show that there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
3. If $\mathbf{b} \notin \mathbf{cone}\, A$, use separation theorems to show that there exists $\mathbf{y} \in \mathbb{R}^m$, such that $\mathbf{A}^\top \mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.
4. Now you can prove Farkas' Lemma: for given $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$, one and only one of the two statements hold:
   - $\exists \mathbf{x} \in \mathbb{R}^n$, $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
   - $\exists \mathbf{y} \in \mathbb{R}^m$, $\mathbf{A}^\top \mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.

Solution: ■