# Introduction to Machine Learning

## Lecture 12: Elementary Reinforcement Learning – Deterministic Environment

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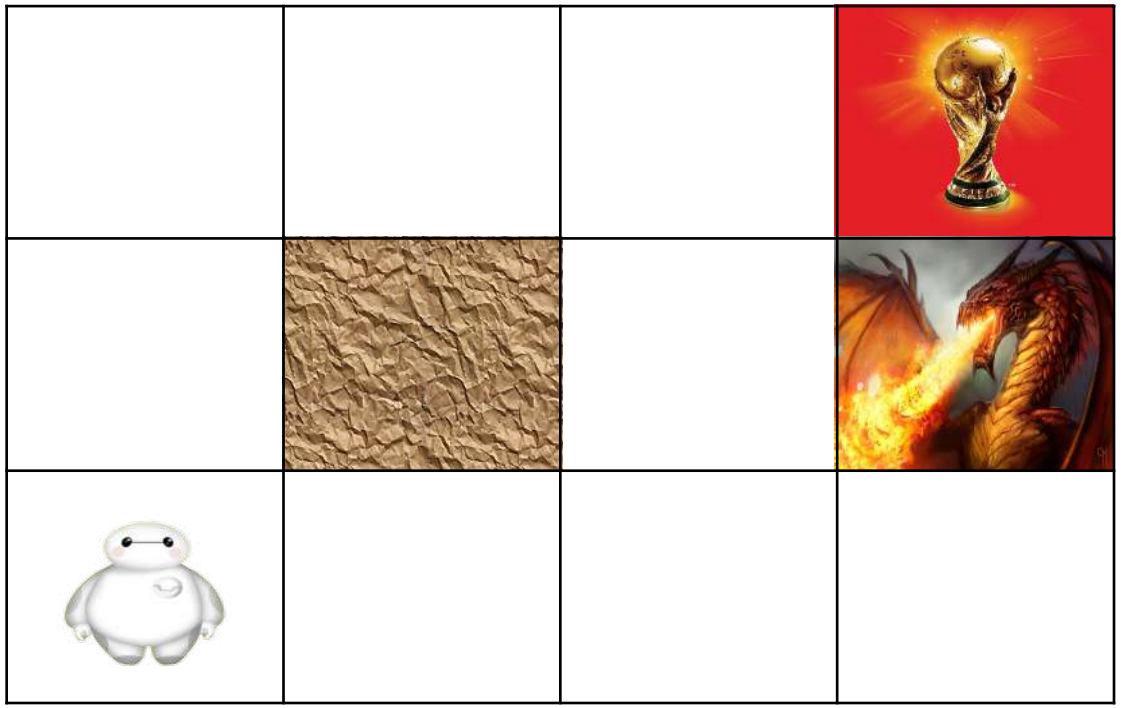
# Learning Scenarios

# Grid World



# Snooker

Ladbrokes
Ladbrokes
Ladbrokes
Ladbrokes
Ladbrokes
优酷
HAWKINS
68
1
0
25
DING
FRI
TV
2016斯诺克巡回赛总决赛第四天五佳球
丁俊晖
5

# Three Kingdoms

Agent

Environment

# Three Kingdoms

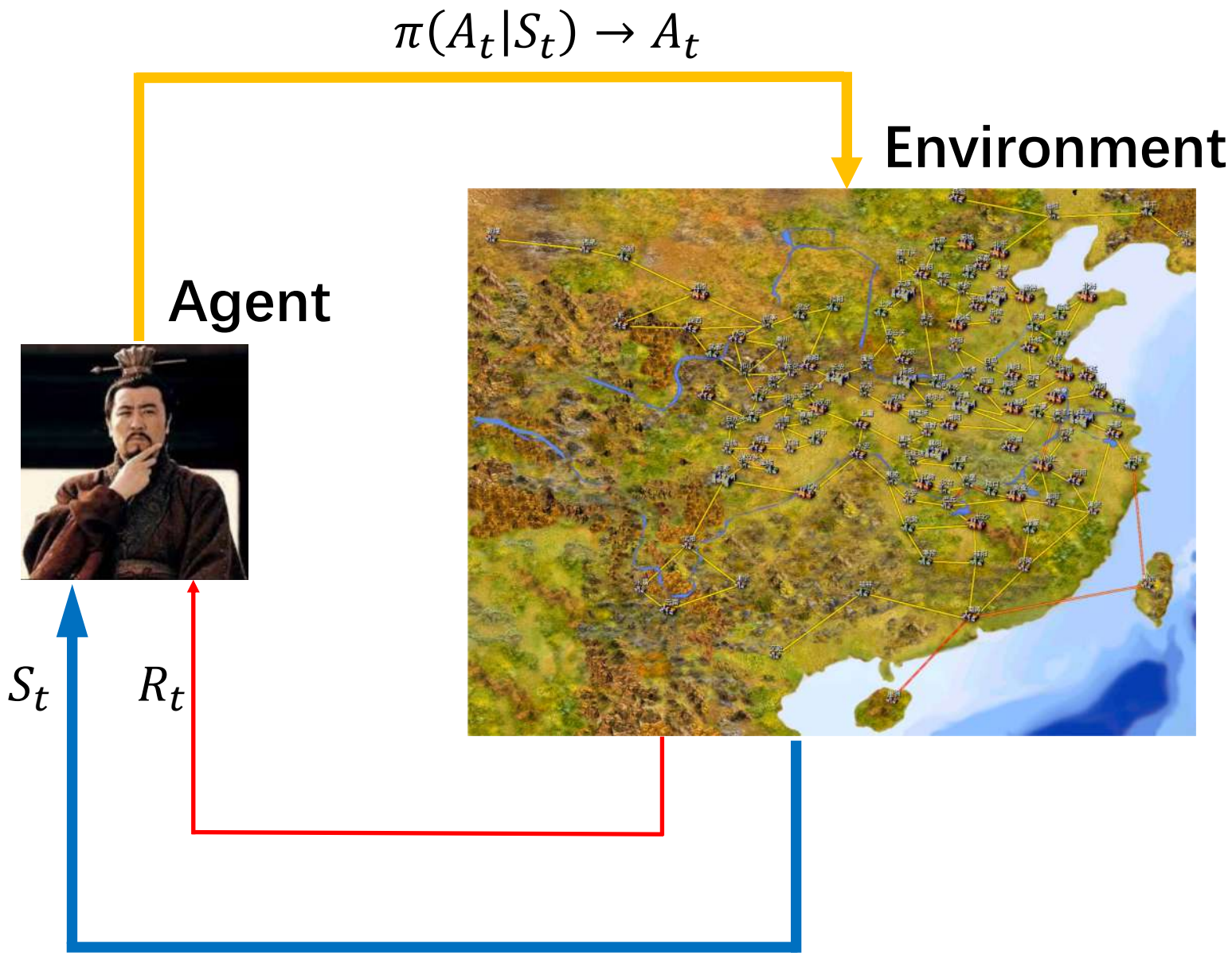
Environment

Agent

$S_t$

$R_t$

# Three Kingdoms



# Three Kingdoms

$\pi(A_t|S_t) \rightarrow A_t$

Environment

Agent

$R_{t+1}$

$S_{t+1}$

$P(S_{t+1}, R_{t+1}|S_t, A_t)$

$\downarrow$

$S_{t+1}, R_{t+1}$

# Three Kingdoms

$\pi(A_t|S_t) \rightarrow A_t$

Environment

Agent

$S_t$

$R_t$

$R_{t+1}$

$S_{t+1}$

$\mathrm{P}(S_{t+1}, R_{t+1}|S_t, A_t)$

$\downarrow$

$S_{t+1}, R_{t+1}$

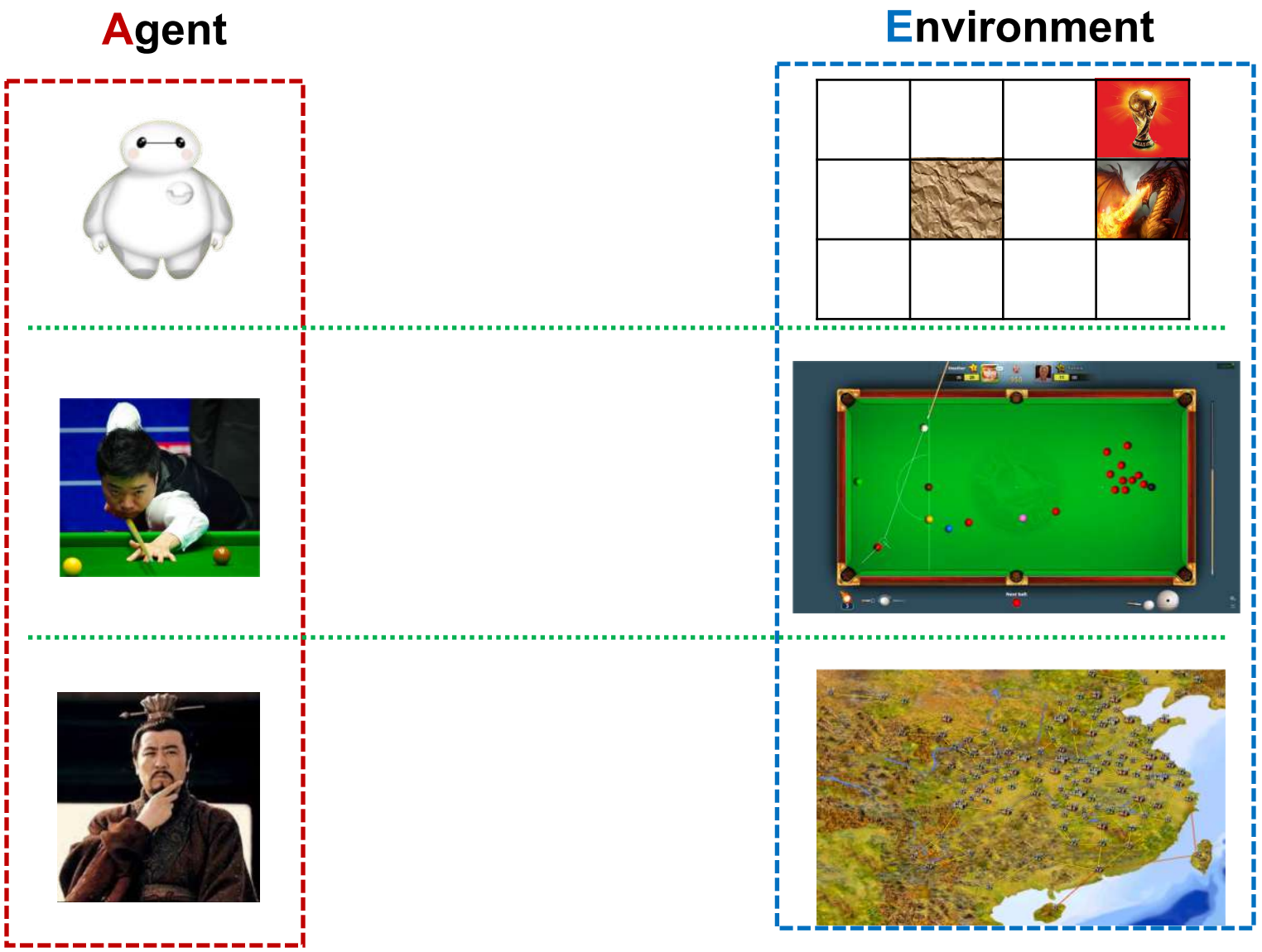
# Agent & Environment

# Agent & Environment

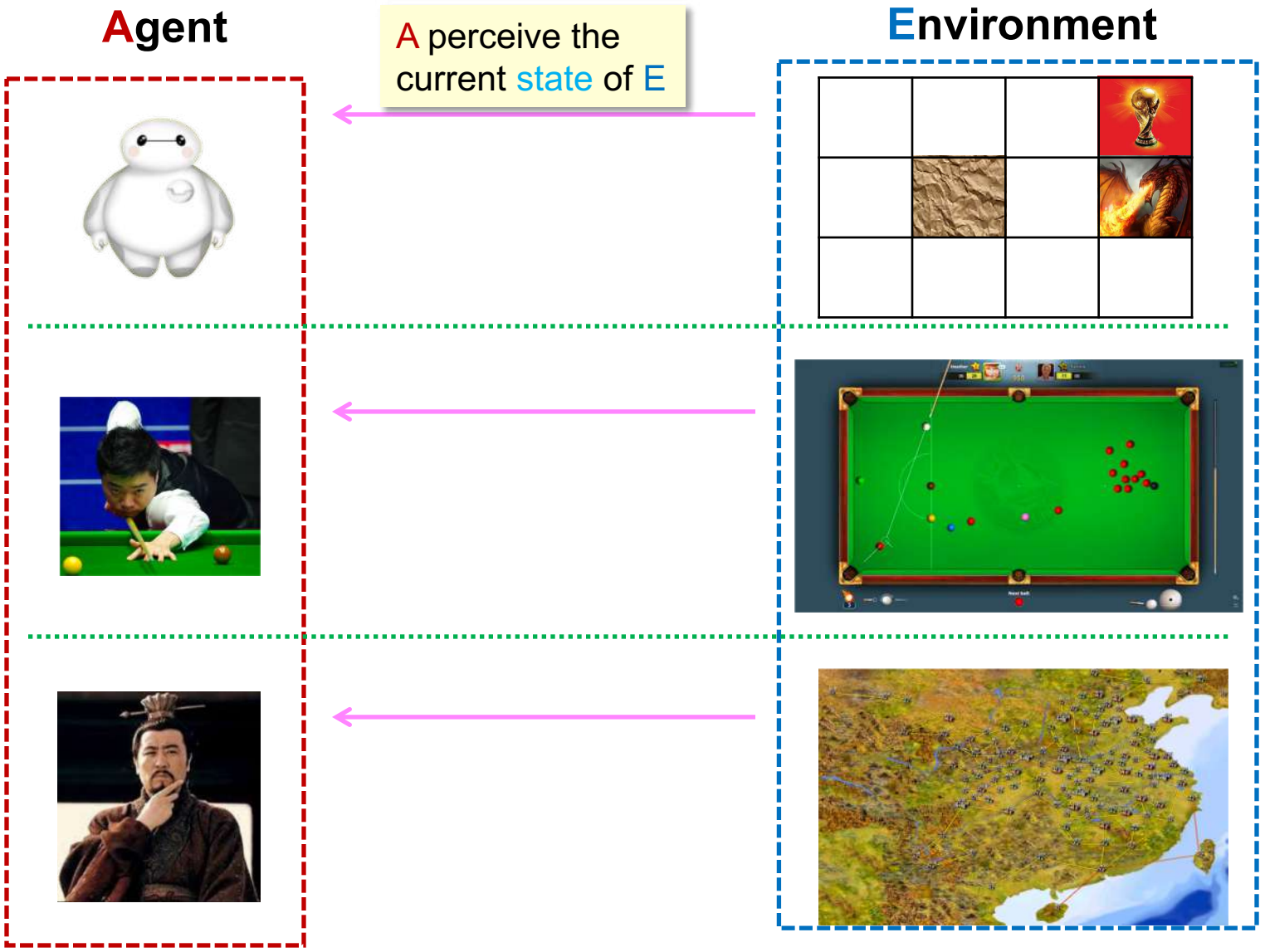
Agent

# Agent & Environment

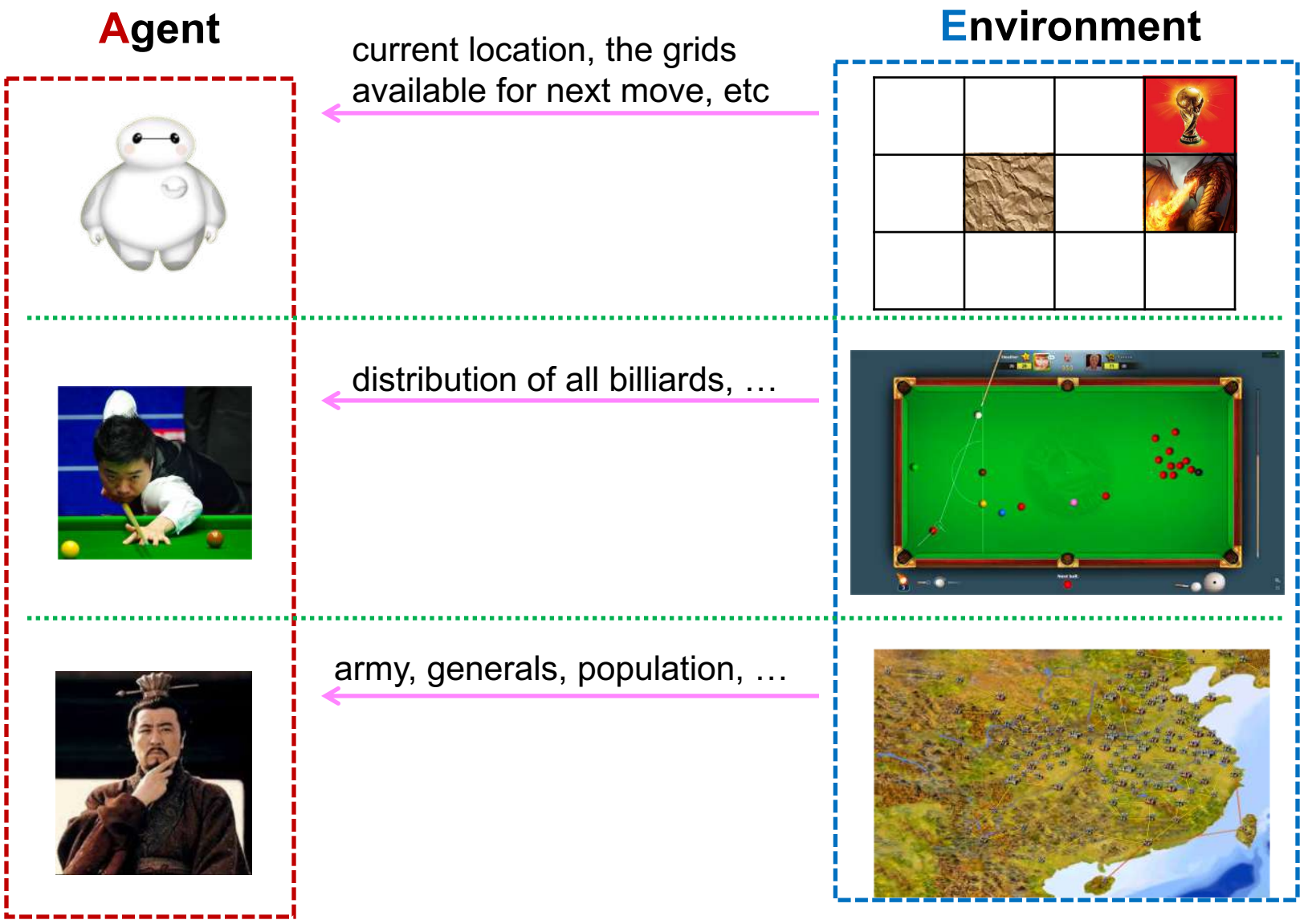
# Agent & Environment



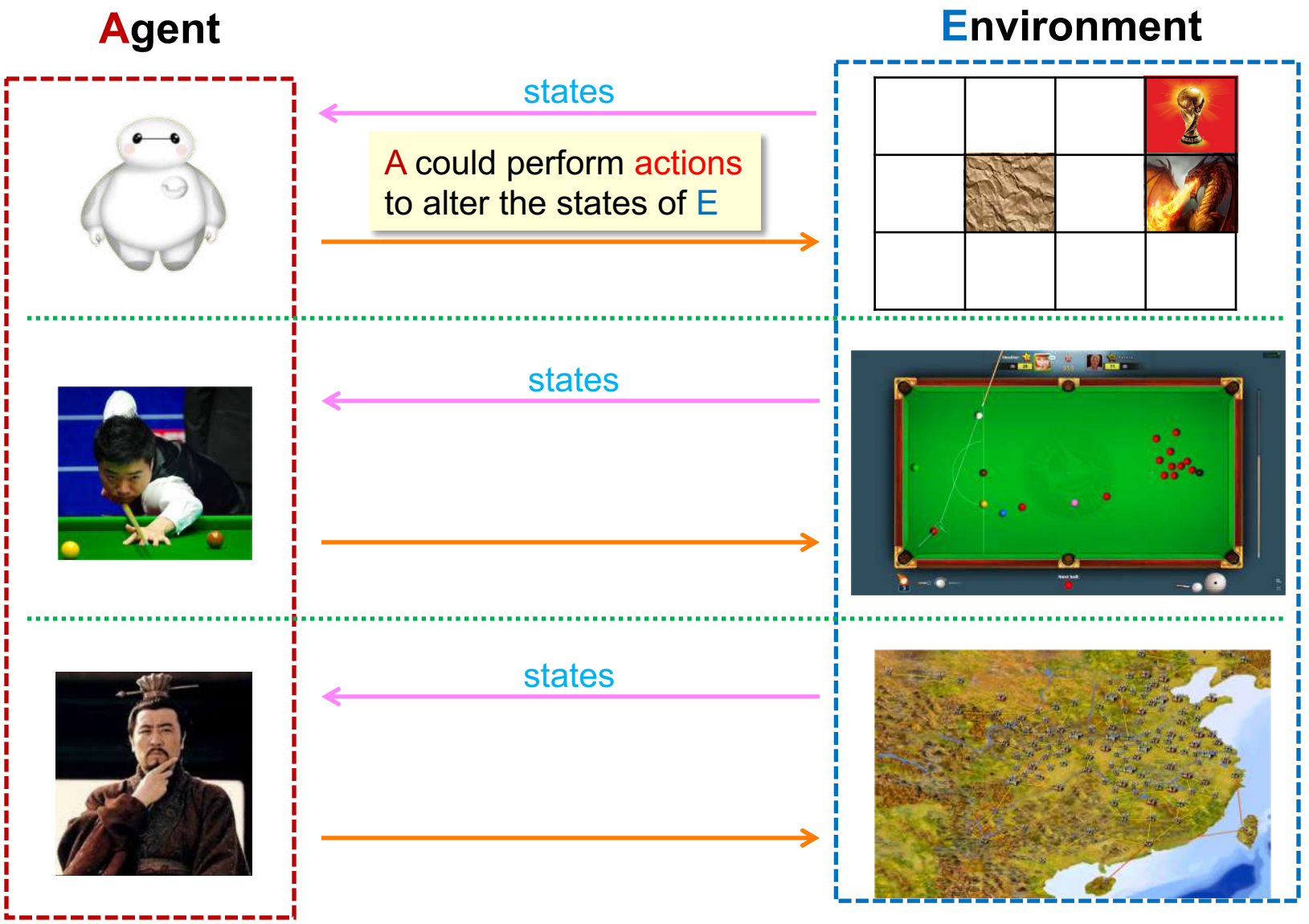
# Interactions between Agent & Environment



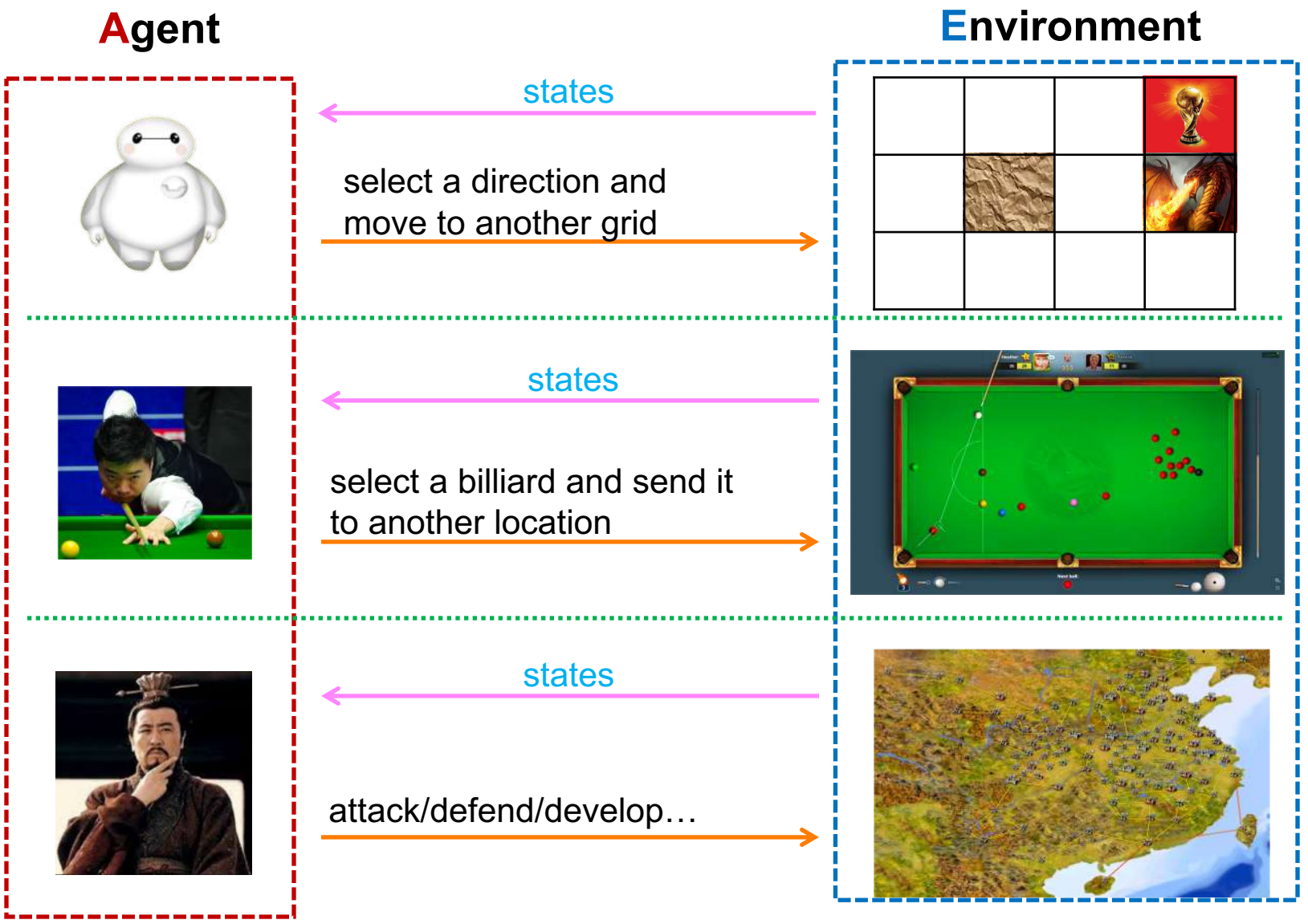
# Interactions between Agent & Environment



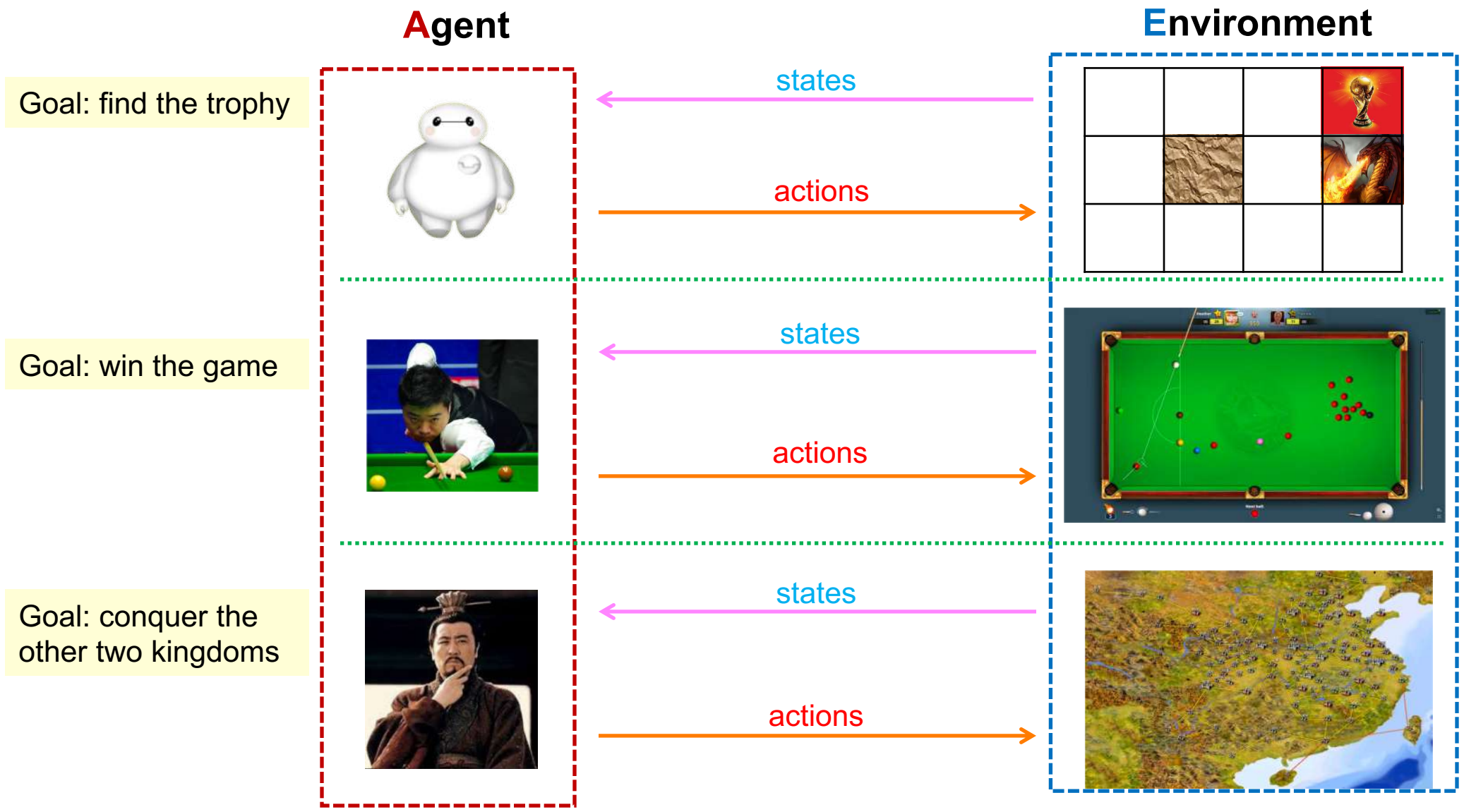
# Interactions between Agent & Environment



# Interactions between Agent & Environment



# Goal State of the Agent



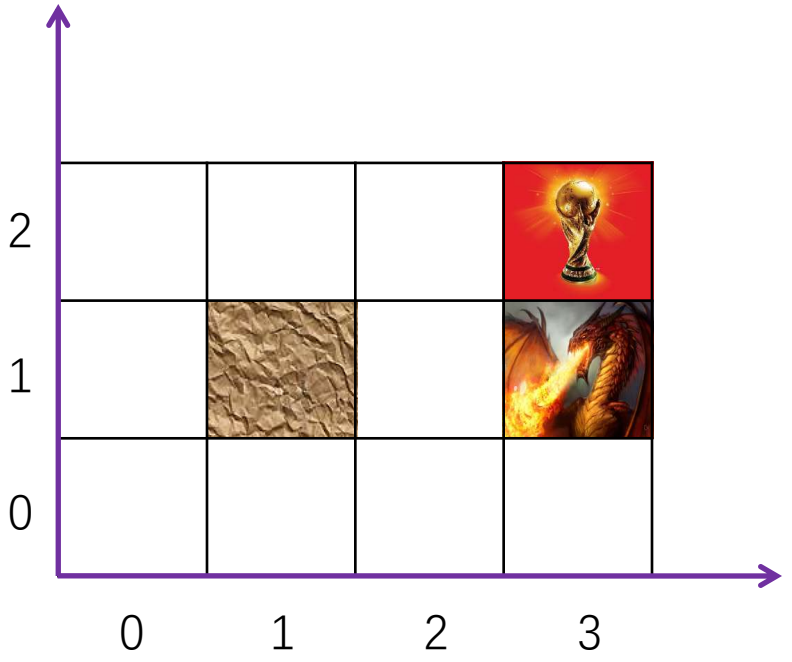
# Markov Decision Process

# Agent & Environment

- The system consists of an agent (may be more) and an environment, interacting with each other.

# States

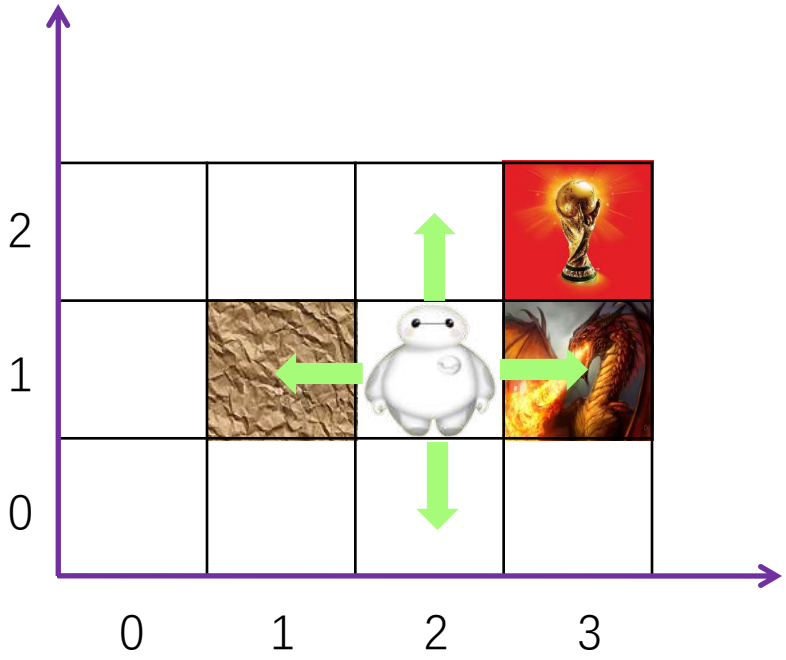
- From the perspective of the agent, the environment is described by a set of states.




States: $\mathcal{S} = \{(i, j) : i = 0, 1, 2, 3, j = 0, 1, 2\}$

# Actions

- At each state, the agent can pick and perform certain action to alter the state.




$$\delta((2,1),\text{up}) = (2,2)$$

$$\delta((2,1),\text{down}) = (2,0)$$

$$\delta((2,1),\text{left}) = (2,1)$$

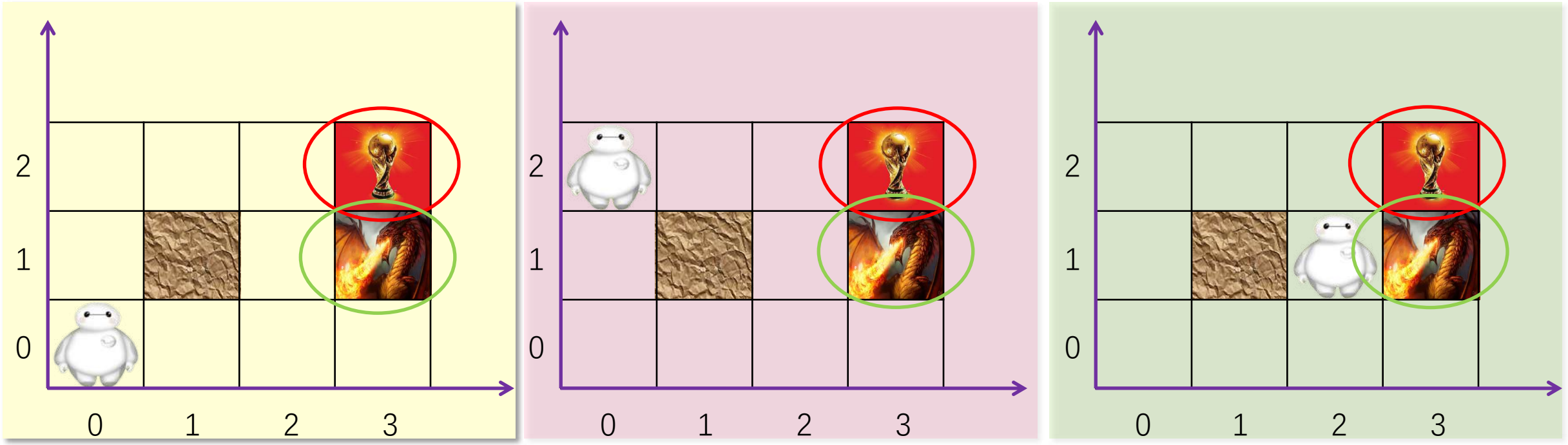
$$\delta((2,1),\text{right}) = (3,1)$$

$\delta : \mathcal{S} \times \mathcal{A} \to \mathcal{S}$ : deterministic environment

Action: $\mathcal{A} = \{\text{up}, \text{down}, \text{left}, \text{right}\}$

# Goal State

- No matter starting from which state, the agent would like to achieve certain goal state.

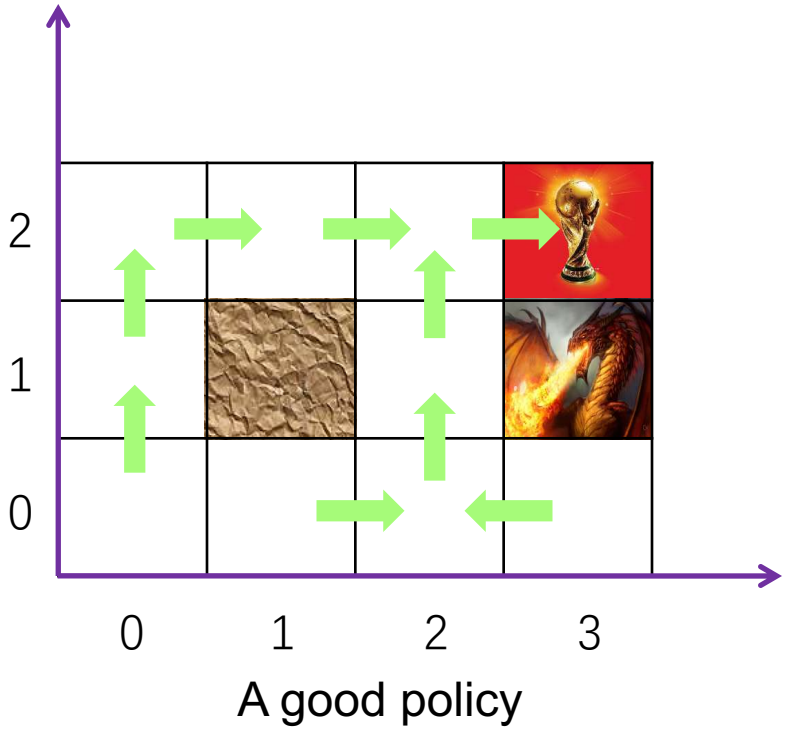



The game will terminate if the agent arrives at $(3, 2)$ (win) or $(3, 1)$ (lose).

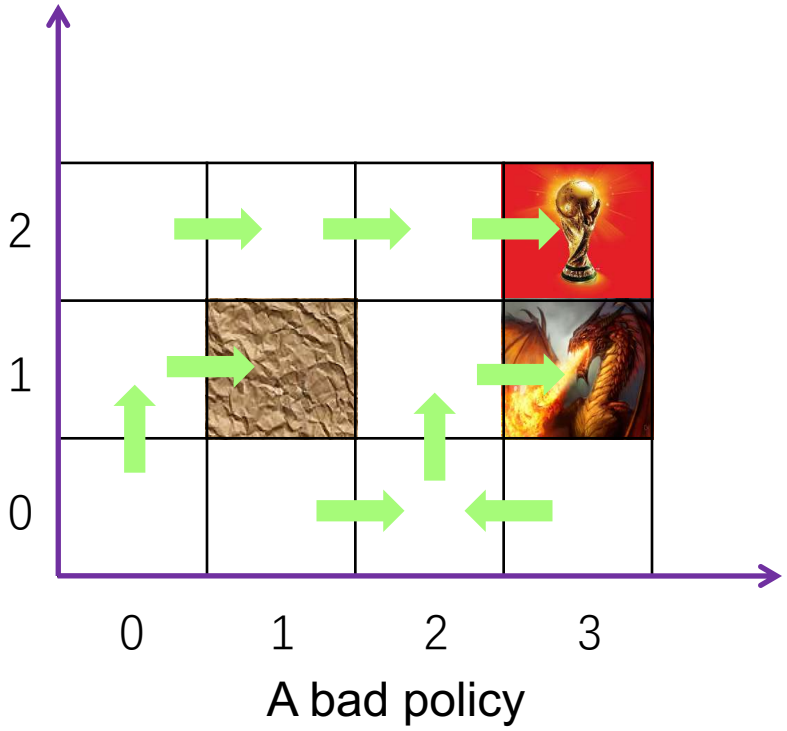
The states $(3, 2)$ and $(3, 1)$ are also called absorbing states

# Policy

- To achieve the goal state, the agent needs to pick and perform a sequence of actions according to the observed states.




A good policy

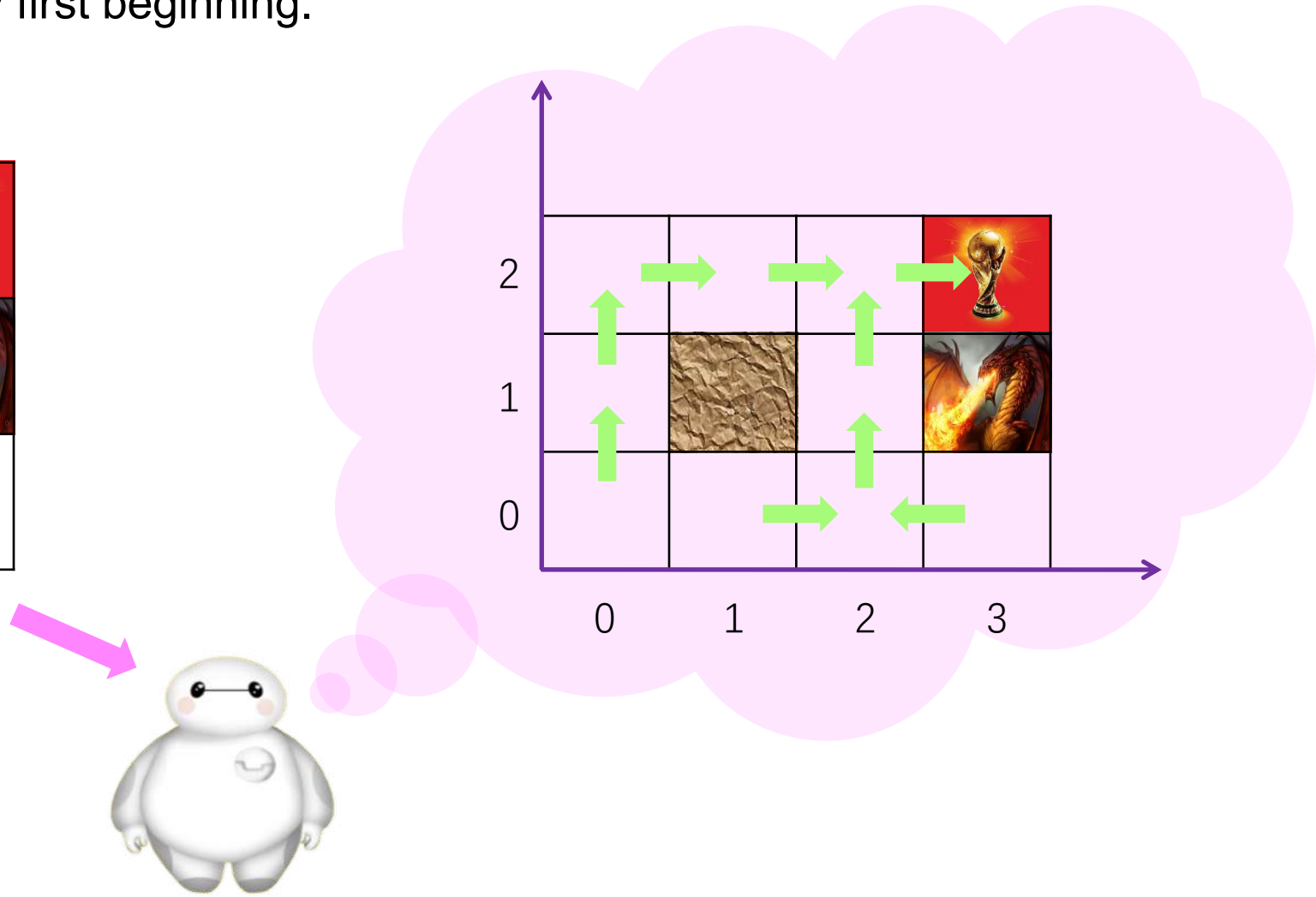



A bad policy

Policy: $\pi : \mathcal{S} \rightarrow \mathcal{A}$

# The Learning Task

- Find a policy that can direct the agent to its goal state no matter which state the agent would have been at the very first beginning.



# The Learning Task

How can we find a desired policy to direct the agent's move?

# Reward

- We assume that the goals of the agent can be defined by a reward function

$$r : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$$

The reward function is not always available. For some applications, you need to define it properly.

- Starting from an arbitrary state, the desired policy would pick for the agent the actions that maximize the reward accumulated over time.

Looking for a policy that would pick for the agent the actions to achieve its goals

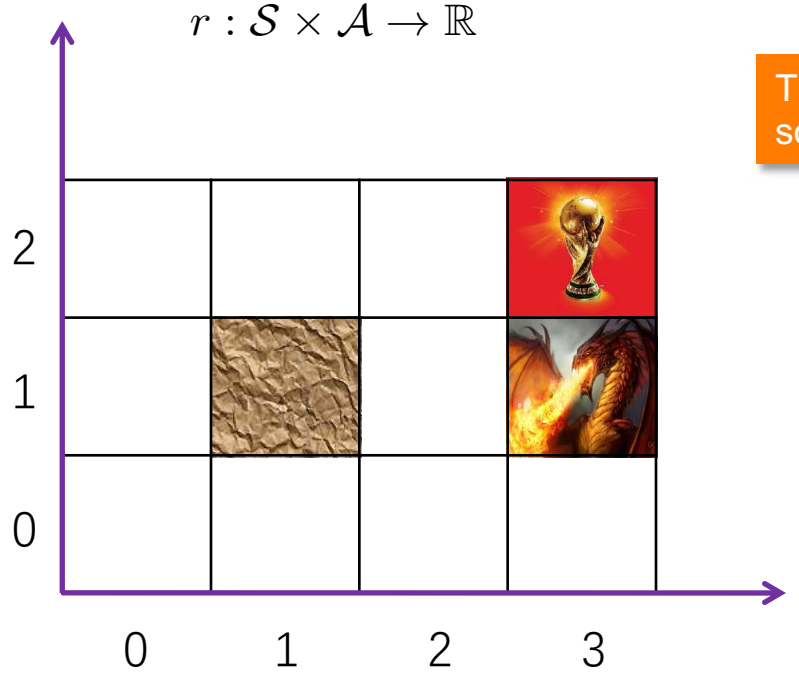
starting from an arbitrary state

Looking for a policy to pick for the agent the actions that will maximize the accumulated reward over time

# Reward

- We assume that the goals of the agent can be defined by a reward function

$$r : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$$




The reward function is not always available. For some applications, you need to define it properly.

$$r(s,a) = \begin{cases} 100, & \text{if } \delta(s,a) = (3,2) \\ -100, & \text{if } \delta(s,a) = (3,1) \\ 0, & \text{otherwise} \end{cases}$$

# Markov Decision Process (MDP)

- Indeed, we have already introduced the so-called MDP, which is defined (rigorously) by
  - a set of states $\mathcal{S}$, possibly infinite
  - a set of actions $\mathcal{A}$, possibly infinite
  - an initial state $s_0 \in \mathcal{S}$
  - a transition probability $\mathbf{P}[s'|s,a]$: distribution over destination states $s' = \delta(s,a)$
  - a reward probability $\mathbf{P}[r|s,a]$: distribution over rewards $r' = r(s,a)$

MRT Chapter 14

- This model is Markovian because the transition and reward probabilities only depend on the current state and the action picked and performed at the current state, instead of the previous sequence of states and actions performed.

- In this lecture, we assume that
  - the states and the actions are finite
  - the environment is **deterministic**, i.e., the destination state and the reward are completely determined by the current state and the action performed at the current state

# The Optimal Policy

Under a MDP, we shall look for the (optimal) policy that leads to the greatest (expected) accumulated reward no matter which state the agent begins with.

# Value Function

- Suppose that a policy $\pi$ is given.

- Starting from an arbitrary state $s_t$, the cumulative reward by following $\pi$ is given by

$$V^{\pi}(s_t) := r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \ldots = \sum_{i=0}^{\infty} \gamma^i r_{t+i}$$

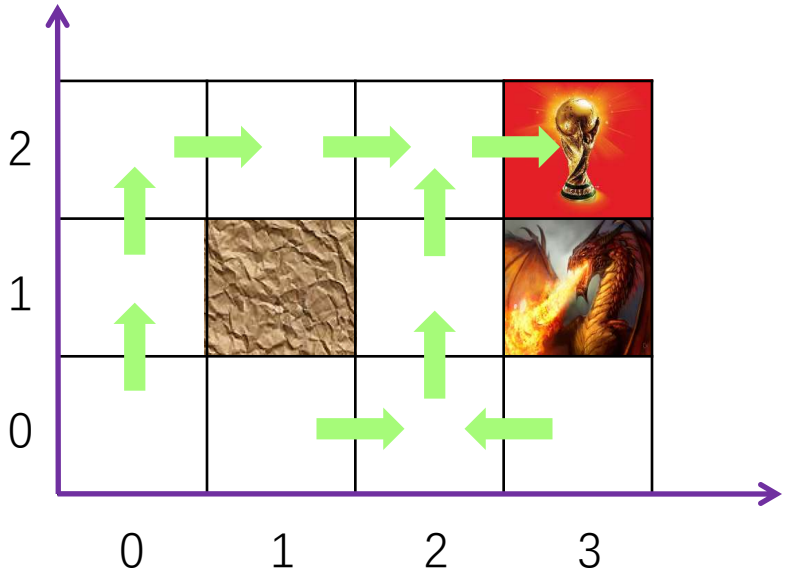
discounted factor, $\gamma \in [0, 1)$

$$s_t \xrightarrow[r_t]{a_t} s_{t+1} \xrightarrow[r_{t+1}]{a_{t+1}} s_{t+2} \xrightarrow[r_{t+2}]{a_{t+2}} \cdots$$

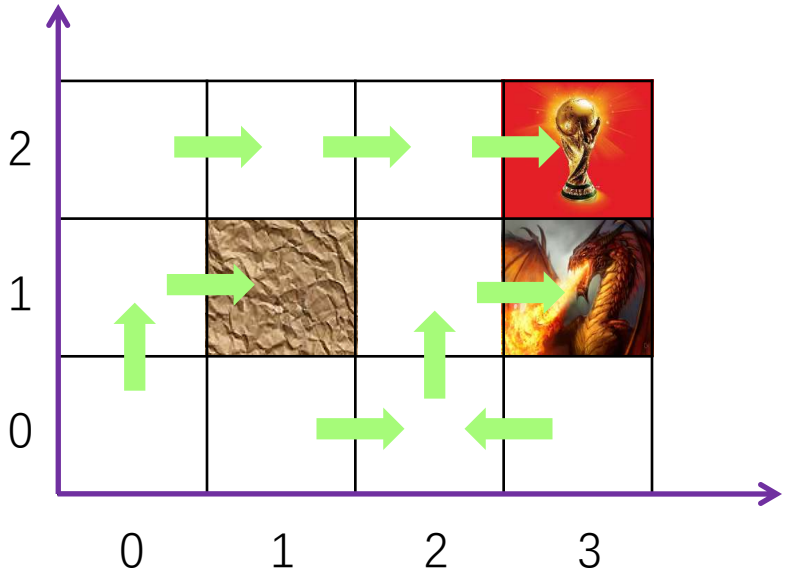
$$a_t = \pi(s_t) \quad r_t = r(s_t, a_t) \quad s_{t+1} = \delta(s_t, a_t)$$

policy reward state transition

# Value Function




A good policy $\pi_1$




A bad policy $\pi_2$

$\gamma = 0.9$

$$V^{\pi_1}((0,0)) = 0.9^4 \times 100 = 65.61$$
$$V^{\pi_1}((1,0)) = 0.9^3 \times 100 = 72.9$$
$$V^{\pi_1}((2,0)) = 0.9^2 \times 100 = 81.0$$
$$V^{\pi_1}((3,0)) = 0.9^3 \times 100 = 72.9$$
$$V^{\pi_1}((0,1)) = 0.9^3 \times 100 = 72.9$$
$$V^{\pi_1}((2,1)) = 0.9 \times 100 = 90.0$$
$$V^{\pi_1}((0,2)) = 0.9^2 \times 100 = 81.0$$
$$V^{\pi_1}((1,2)) = 0.9 \times 100 = 90.0$$
$$V^{\pi_1}((2,2)) = 100.0$$

$$V^{\pi_2}((0,0)) = 0$$
$$V^{\pi_2}((1,0)) = 0.9^2 \times (-100) = -81.0$$
$$V^{\pi_2}((2,0)) = 0.9 \times (-100) = -90.0$$
$$V^{\pi_2}((3,0)) = 0.9^2 \times (-100) = -81$$
$$V^{\pi_2}((0,1)) = 0$$
$$V^{\pi_2}((2,1)) = -100.0$$
$$V^{\pi_2}((0,2)) = 0.9^2 \times 100 = 81.0$$
$$V^{\pi_2}((1,2)) = 0.9 \times 100 = 90.0$$
$$V^{\pi_2}((2,2)) = 100.0$$

# Value Function – Bellman Equation

- Starting from an arbitrary state $s_t$, the cumulative reward by following $\pi$ is given by

$$V^{\pi}(s_t) := r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \ldots = \sum_{i=0}^{\infty} \gamma^i r_{t+i}$$

$$s_t \xrightarrow[r_t]{a_t} s_{t+1} \xrightarrow[r_{t+1}]{a_{t+1}} s_{t+2} \xrightarrow[r_{t+2}]{a_{t+2}} \cdots$$

$$a_t = \pi(s_t) \quad r_t = r(s_t, a_t) \quad s_{t+1} = \delta(s_t, a_t)$$

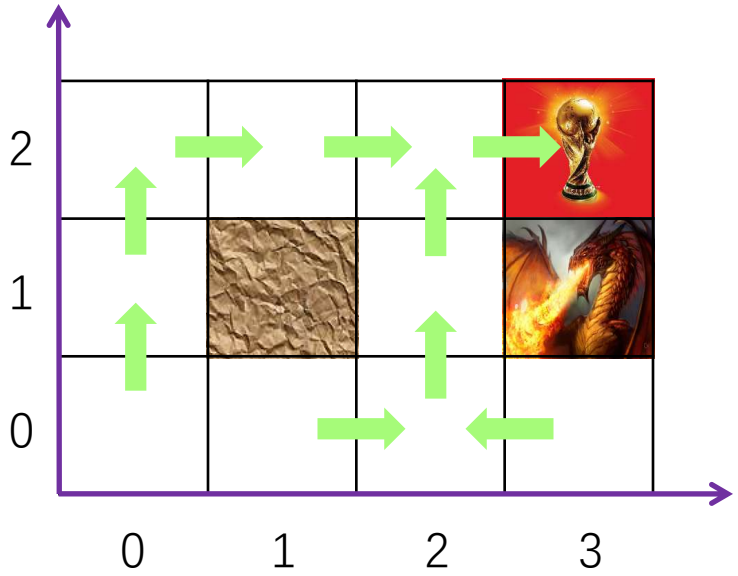
- **Bellman Equation**

$$\begin{aligned} V^{\pi}(s_t) &= r_t + \gamma \left(r_{t+1} + \gamma r_{t+2} + \ldots\right) \\ &= r_t + \gamma V^{\pi}(s_{t+1}) \\ &= r_t + \gamma V^{\pi}(\delta(s_t, a_t)) \end{aligned}$$

# Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = r(s,a) + \gamma V^{\pi}(\delta(s,a))$$



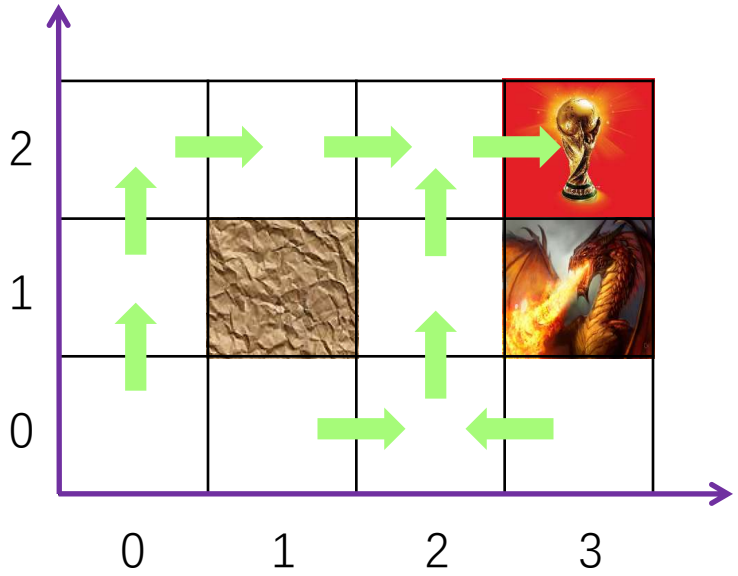

A good policy $\pi_1$

$$\begin{pmatrix} V^{\pi_1}((0,0)) \\ V^{\pi_1}((1,0)) \\ V^{\pi_1}((2,0)) \\ V^{\pi_1}((3,0)) \\ V^{\pi_1}((0,1)) \\ V^{\pi_1}((1,1)) \\ V^{\pi_1}((2,1)) \\ V^{\pi_1}((3,1)) \\ V^{\pi_1}((0,2)) \\ V^{\pi_1}((1,2)) \\ V^{\pi_1}((2,2)) \\ V^{\pi_1}((3,2)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 100 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^{\pi_1}((0,0)) \\ V^{\pi_1}((1,0)) \\ V^{\pi_1}((2,0)) \\ V^{\pi_1}((3,0)) \\ V^{\pi_1}((0,1)) \\ V^{\pi_1}((1,1)) \\ V^{\pi_1}((2,1)) \\ V^{\pi_1}((3,1)) \\ V^{\pi_1}((0,2)) \\ V^{\pi_1}((1,2)) \\ V^{\pi_1}((2,2)) \\ V^{\pi_1}((3,2)) \end{pmatrix}$$

# Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = r(s,a) + \gamma V^{\pi}(\delta(s,a))$$

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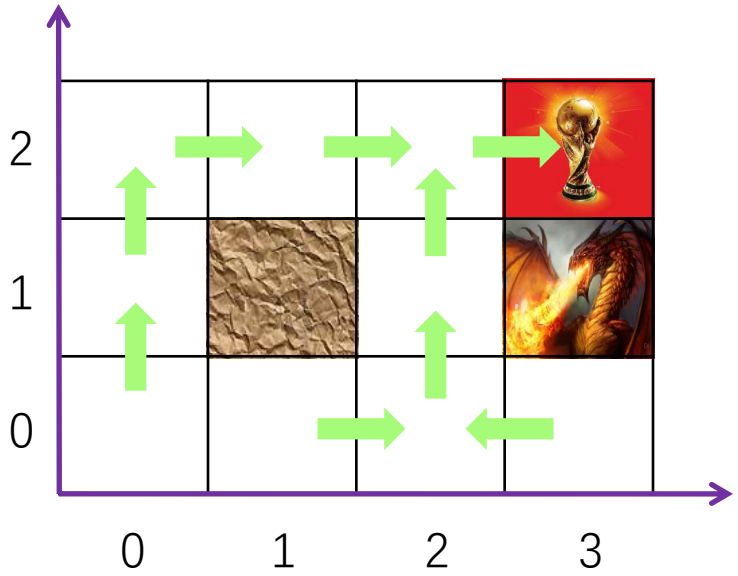
# Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = r(s,a) + \gamma V^{\pi}(\delta(s,a))$$




A good policy $\pi_1$

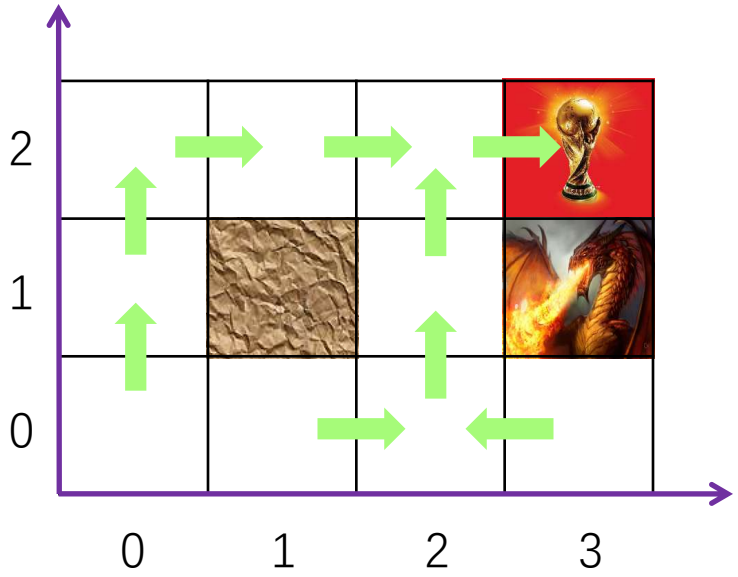
$$\begin{pmatrix} V^{\pi_1}((0,0)) \\ V^{\pi_1}((1,0)) \\ V^{\pi_1}((2,0)) \\ V^{\pi_1}((3,0)) \\ V^{\pi_1}((0,1)) \\ V^{\pi_1}((1,1)) \\ V^{\pi_1}((2,1)) \\ V^{\pi_1}((3,1)) \\ V^{\pi_1}((0,2)) \\ V^{\pi_1}((1,2)) \\ V^{\pi_1}((2,2)) \\ V^{\pi_1}((3,2)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 100 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^{\pi_1}((0,0)) \\ V^{\pi_1}((1,0)) \\ V^{\pi_1}((2,0)) \\ V^{\pi_1}((3,0)) \\ V^{\pi_1}((0,1)) \\ V^{\pi_1}((1,1)) \\ V^{\pi_1}((2,1)) \\ V^{\pi_1}((3,1)) \\ V^{\pi_1}((0,2)) \\ V^{\pi_1}((1,2)) \\ V^{\pi_1}((2,2)) \\ V^{\pi_1}((3,2)) \end{pmatrix}$$

$$V = R + \gamma T V$$

# Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = r(s,a) + \gamma V^{\pi}(\delta(s,a))$$




A good policy $\pi_1$

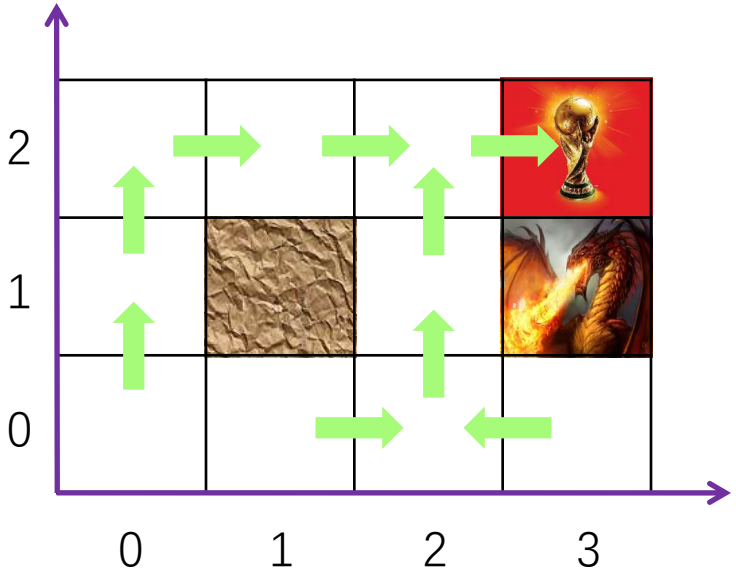
$$V = R + \gamma T V$$

$$V = (I - \gamma T)^{-1} R$$

# Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = r(s,a) + \gamma V^{\pi}(\delta(s,a))$$




A good policy $\pi_1$

$$V = R + \gamma T V$$

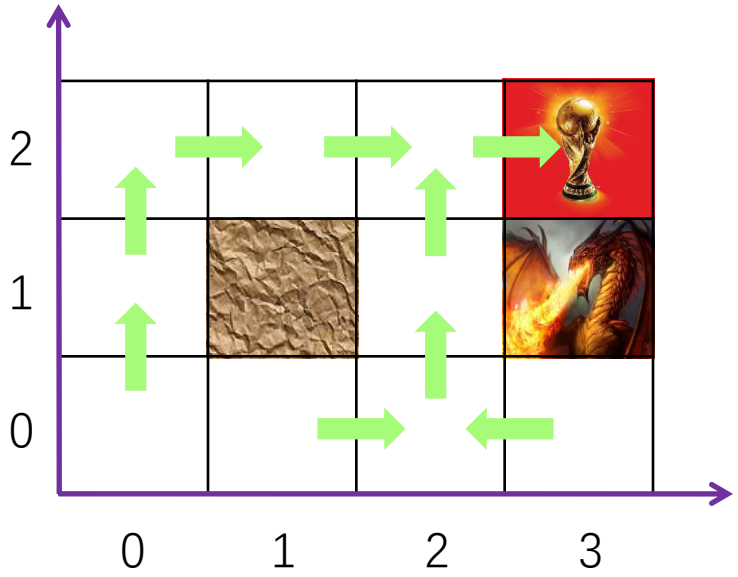
$$V = (I - \gamma T)^{-1} R$$

invertible ?

# Value Function – Bellman Equation

- **Bellman Equation**

$$V^{\pi}(s) = r(s,a) + \gamma V^{\pi}(\delta(s,a))$$




A good policy $\pi_1$

***Theorem:*** *For a finite MDP, Bellman's equation admits a unique solution that is given by*

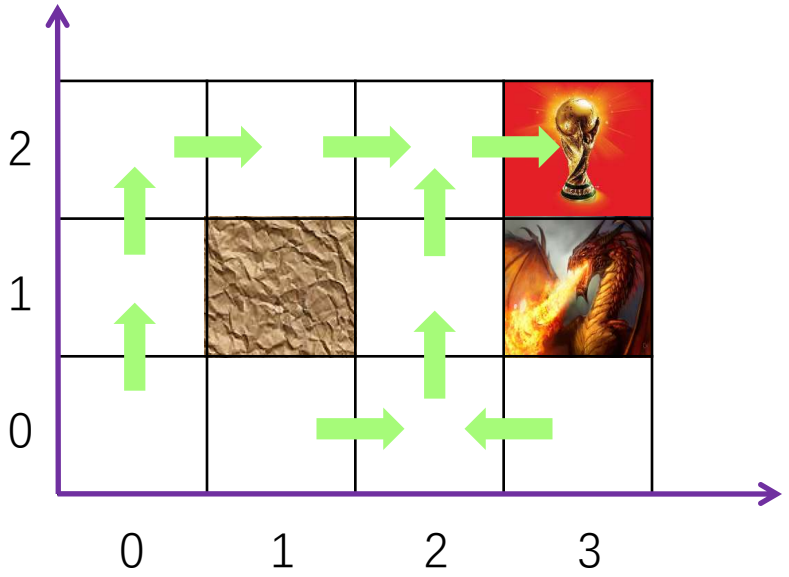
$$V = (I - \gamma T)^{-1} R$$

- The vector $R$ and matrix $T$ depend on the policy

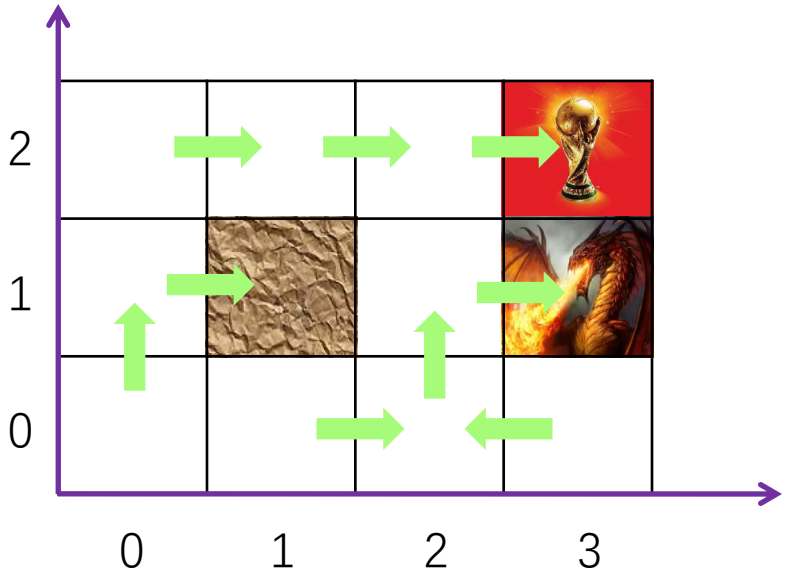
# The Learning Task Revisited

- The learning task for RL scenarios is to learn an optimal policy in the sense that

$$\pi^* := \text{argmax}_\pi \, V^\pi(s), \, \forall \, s.$$




A good policy $\pi_1$




A bad policy $\pi_2$

- For $\pi_1$ and $\pi_2$, we have

$$V^{\pi_1}(s) \geq V^{\pi_2}(s), \, \forall \, s.$$

- Indeed, $\pi_1$ is the optimal policy.

# The Q Function

- Learning the optimal policy is challenging
- An alternative approach to find the optimal policy indirectly is by computing the state-action value function (Q function)

$$Q(s,a) = r(s,a) + \gamma V^*(\delta(s,a))$$

$Q(s,a)$ is the accumulated reward by performing the action $a$ first and then following the optimal policy

- The definition of the optimal policy implies that

$$\pi^*(s) = \text{argmax}_a\, Q(s,a) = \text{argmax}_a\, r(s,a) + \gamma V^*(\delta(s,a))$$

- Notice that

$$V^*(s) = \max_a\, Q(s,a) = \max_a\, r(s,a) + \gamma V^*(\delta(s,a))$$

- All together, we have

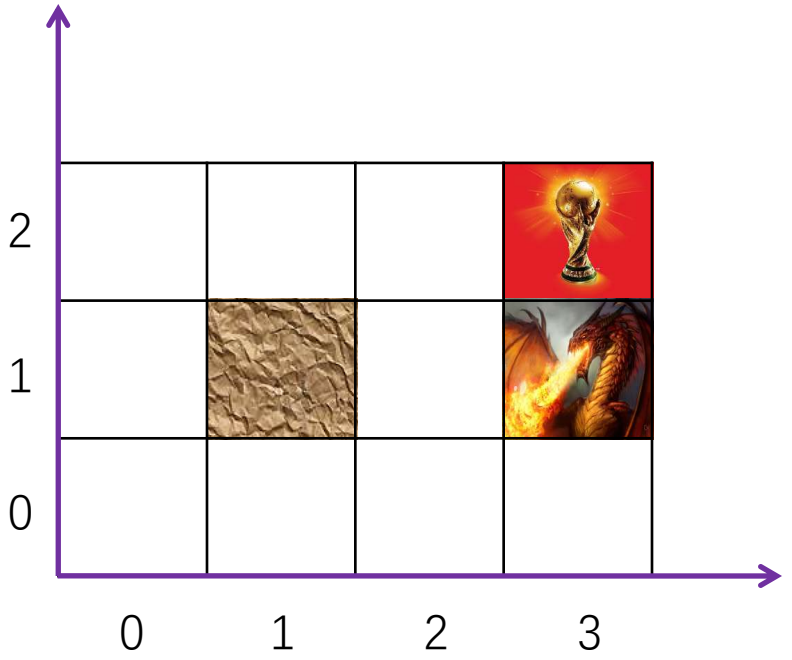
$$Q(s,a) = r(s,a) + \gamma \max_{a'} Q(\delta(s,a), a')$$

Bellman Equations for the optimal policy

# Planning Algorithms

# Planning

- Planning: to find the optimal policy, there is no need for the agent to actually perform actions and interact with the environment




**Known**

$\delta : \mathcal{S} \times \mathcal{A} \to \mathcal{S}$ : state transition

$r : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ : reward

# Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy

**Initialize** $V(s)$ to arbitrary values

**while** termination conditions does not hold

  **For** $s \in \mathcal{S}$

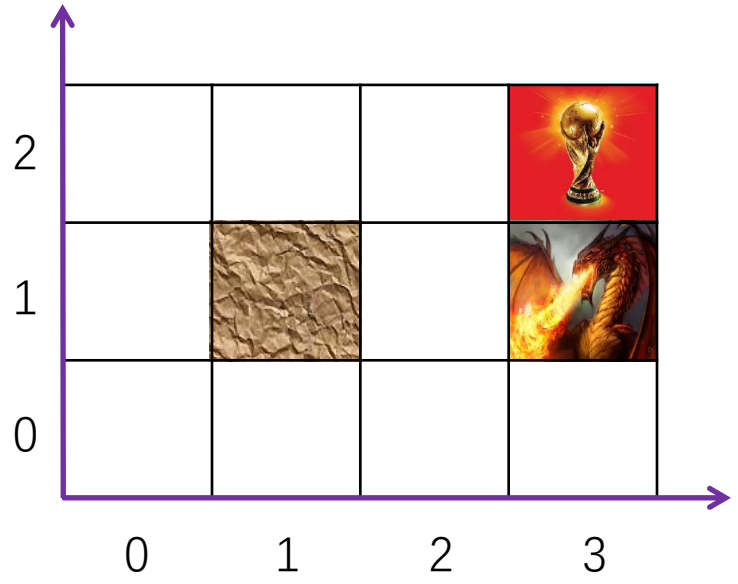
    **For** $a \in \mathcal{A}$

      $Q(s,a) \leftarrow r(s,a) + \gamma V(\delta(s,a))$

  $V(s) \leftarrow \max_a Q(s,a)$

# Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy

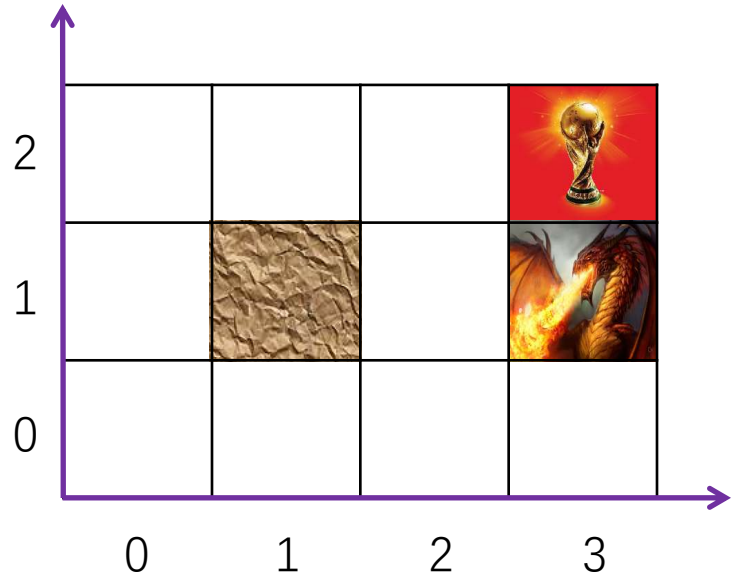



Example

$$V \leftarrow 0$$
$$Q((0,0), \text{up}) \leftarrow 0 + 0.9 \times V((0,1)) = 0$$
$$Q((0,0), \text{down}) \leftarrow 0 + 0.9 \times V((0,0)) = 0$$
$$Q((0,0), \text{left}) \leftarrow 0 + 0.9 \times V((0,0)) = 0$$
$$Q((0,0), \text{right}) \leftarrow 0 + 0.9 \times V((1,0)) = 0$$
$$V((0,0)) \leftarrow \max\{Q((0,0), \text{up}), Q((0,0), \text{down}), Q((0,0), \text{left}), Q((0,0), \text{right})\} = 0$$

# Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy



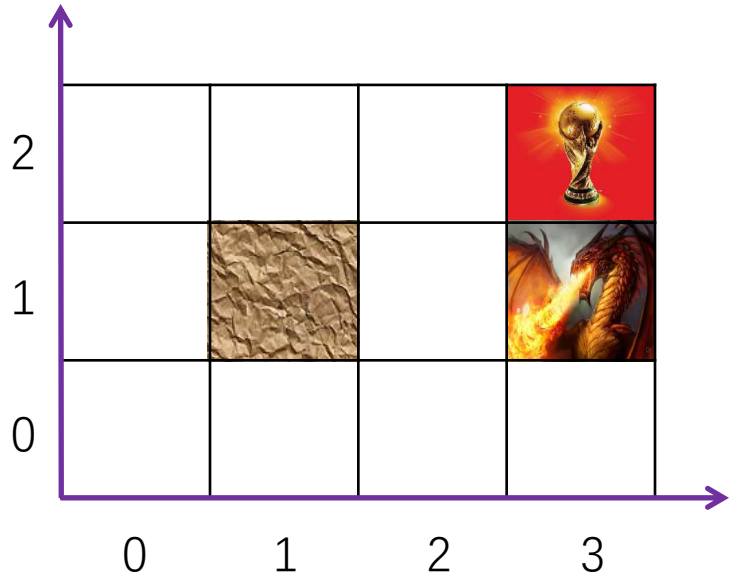

Example

$$V \leftarrow 0$$
$$Q((0,0), \text{up}) \leftarrow 0 + 0.9 \times V((0,1)) = 0$$
$$Q((0,0), \text{down}) \leftarrow 0 + 0.9 \times V((0,0)) = 0$$
$$Q((0,0), \text{left}) \leftarrow 0 + 0.9 \times V((0,0)) = 0$$
$$Q((0,0), \text{right}) \leftarrow 0 + 0.9 \times V((1,0)) = 0$$
$$V((0,0)) \leftarrow \max\{Q((0,0), \text{up}), Q((0,0), \text{down}), Q((0,0), \text{left}), Q((0,0), \text{right})\} = 0$$

Nothing happens

# Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy




Example

$$V \leftarrow 0$$
$$Q((2,2),\text{up}) \leftarrow 0 + 0.9 \times V((2,2)) = 0$$
$$Q((2,2),\text{down}) \leftarrow 0 + 0.9 \times V((2,1)) = 0$$
$$Q((2,2),\text{left}) \leftarrow 0 + 0.9 \times V((1,2)) = 0$$
$$Q((2,2),\text{right}) \leftarrow 100 + 0.9 \times V((3,2)) = 100$$
$$V((2,2)) \leftarrow \max\{Q((2,2),\text{up}), Q((2,2),\text{down}), Q((2,2),\text{left}), Q((2,2),\text{right})\} = 100$$

# Value Iteration

- Value iteration aims to find the optimal value function and thus the optimal policy

> ***Theorem:*** *For any initial value $V$, the sequence generated by the value iteration algorithm converges to $V^*$.*

  - The key to the proof is the contraction mapping theorem

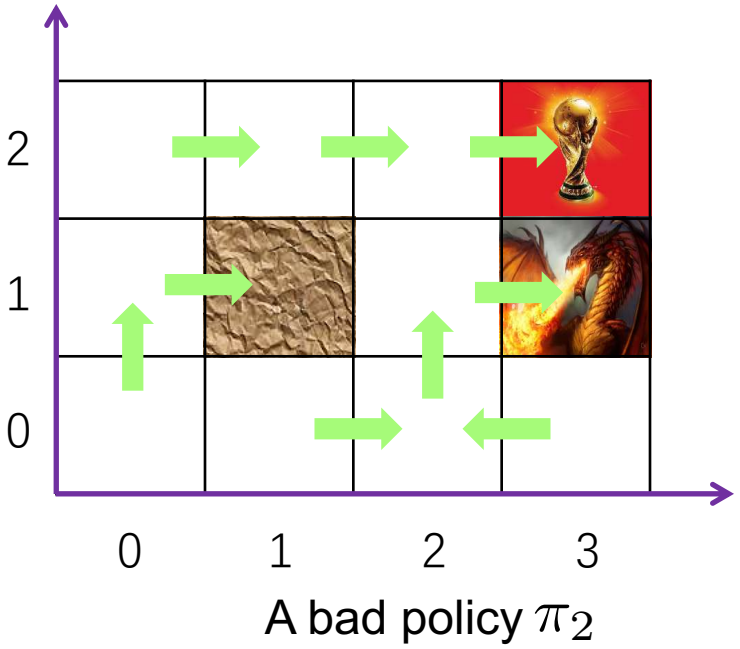
# Policy Iteration

- Policy iteration improves the policy directly

$$
\begin{aligned}
&\textbf{Initialize } \pi, \pi' \text{ to two different policies} \\
&\textbf{while}(\pi \neq \pi') \\
&\quad V \leftarrow (I - \gamma T^{\pi})^{-1} R^{\pi} \\
&\quad \pi' \leftarrow \pi \\
&\quad\quad \textbf{For } s \in \mathcal{S} \\
&\quad\quad \pi(s) \leftarrow \text{argmax}_a \, r(s,a) + \gamma V(\delta(s,a))
\end{aligned}
$$

# Policy Iteration

- Policy iteration improves the policy directly



A bad policy $\pi_2$

**Initialize** $\pi \leftarrow \pi_2, \pi' \neq \pi_2$

**while**$(\pi \neq \pi')$

$1^{st}$ $V \leftarrow (I - \gamma T^{\pi})^{-1} R^{\pi}$

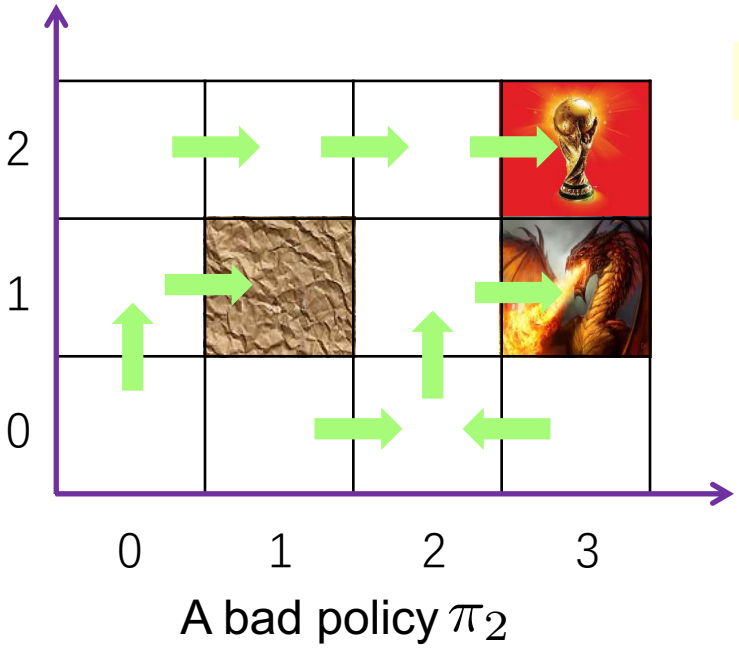
$\pi' \leftarrow \pi$

**For** $s \in \mathcal{S}$

$\pi(s) \leftarrow \text{argmax}_a \, r(s,a) + \gamma V(\delta(s,a))$

# Policy Iteration

- Policy iteration improves the policy directly




A bad policy $\pi_2$

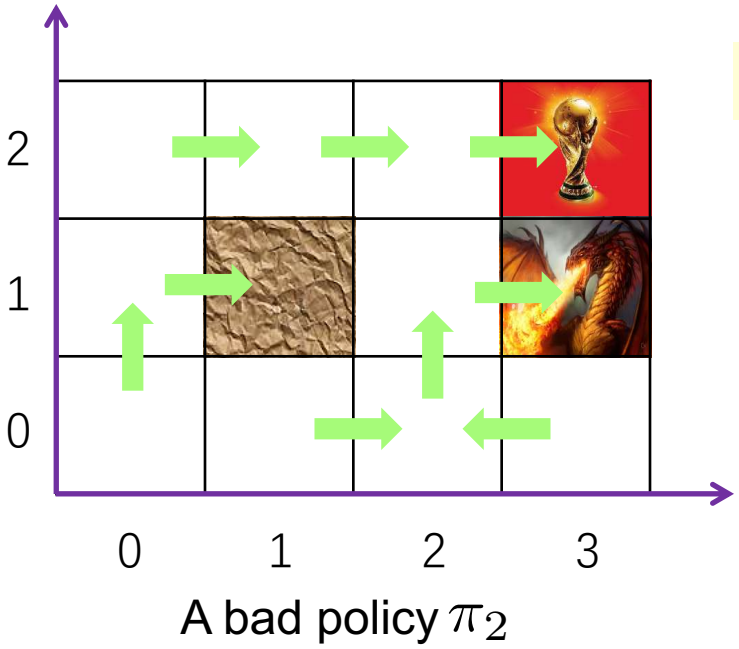
$1^{st}$ iteration

$$V^{\pi} = \begin{pmatrix} V^{\pi}(0,0) \\ V^{\pi}(1,0) \\ V^{\pi}(2,0) \\ V^{\pi}(3,0) \\ V^{\pi}(0,1) \\ V^{\pi}(2,1) \\ V^{\pi}(3,1) \\ V^{\pi}(0,2) \\ V^{\pi}(1,2) \\ V^{\pi}(2,2) \\ V^{\pi}(3,2) \end{pmatrix}$$

11 states in total

# Policy Iteration

- Policy iteration improves the policy directly



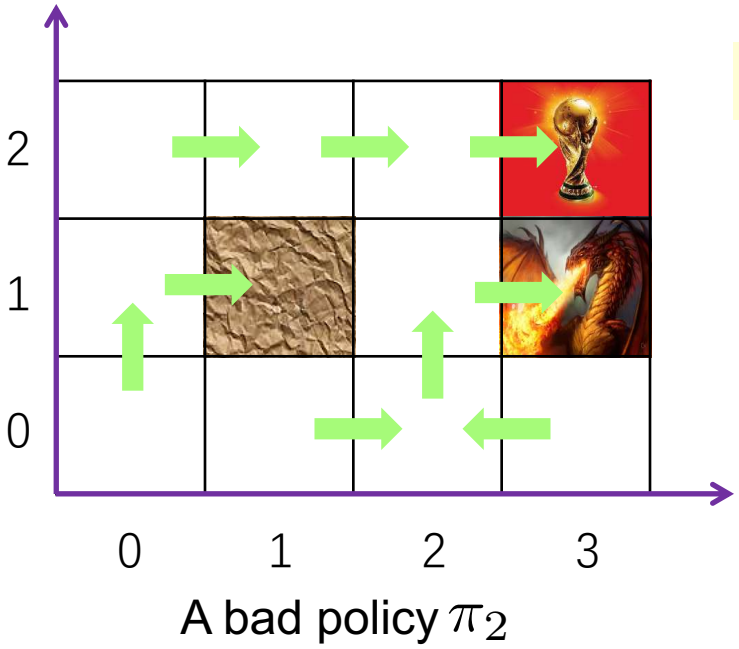
A bad policy $\pi_2$

$1^{st}$ iteration

$$R^{\pi} = \begin{pmatrix} r((0,0),\text{up}) \\ r((1,0),\text{right}) \\ r((2,0),\text{up}) \\ r((3,0),\text{left}) \\ r((0,1),\text{right}) \\ r((2,1),\text{right}) \\ r((3,1),\text{END}) \\ r((0,2),\text{right}) \\ r((1,2),\text{right}) \\ r((2,2),\text{right}) \\ r((3,2),\text{END}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ 0 \\ 100 \\ 0 \end{pmatrix}$$

# Policy Iteration

- Policy iteration improves the policy directly



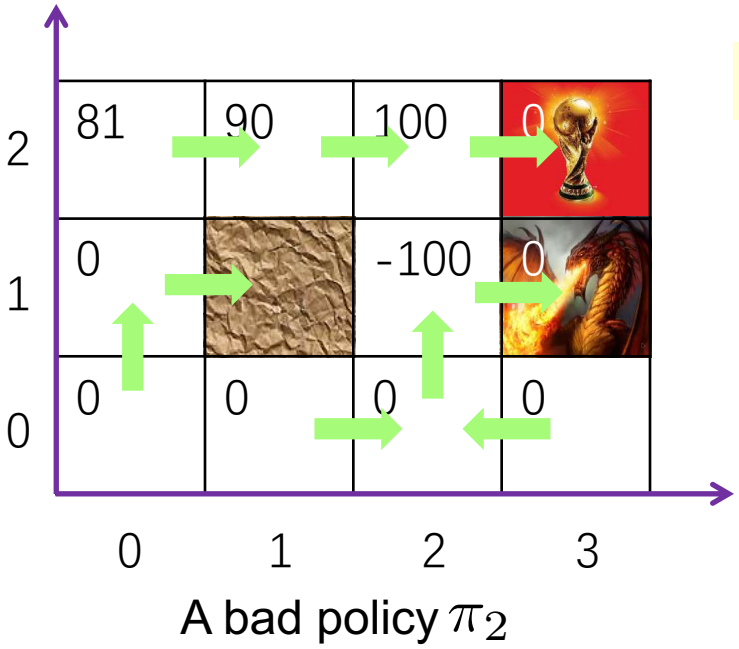

A bad policy $\pi_2$

$1^{st}$ iteration

$$T^{\pi} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

# Policy Iteration

- Policy iteration improves the policy directly



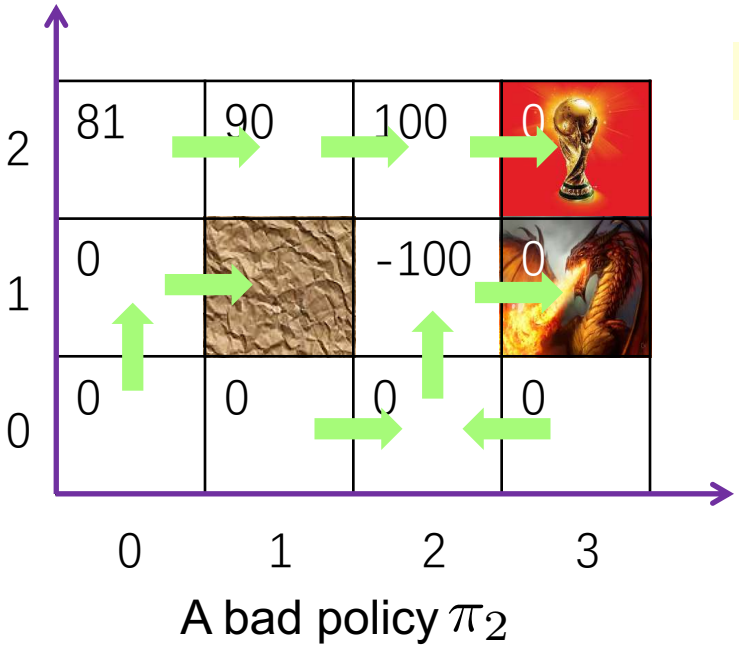

A bad policy $\pi_2$

$1^{st}$ iteration

$$V \leftarrow (I - \gamma T^{\pi})^{-1} R^{\pi} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -100 \\ 0 \\ 81 \\ 90 \\ 100 \\ 0 \end{pmatrix}$$

# Policy Iteration

- Policy iteration improves the policy directly




A bad policy $\pi_2$

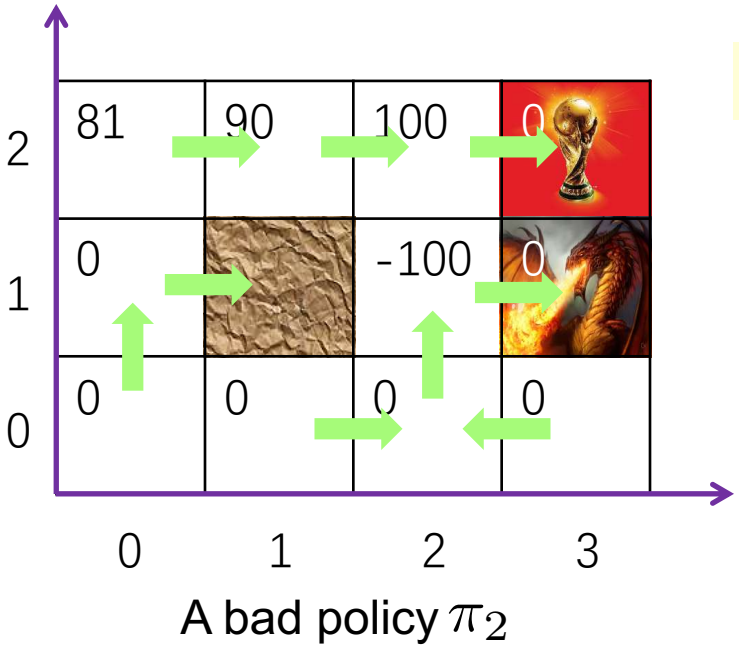
$1^{st}$ iteration: update the policy

$$\pi((0,0)) = \text{argmax}_a\{r((0,0),\text{up}) + \gamma V((0,1)),$$
$$r((0,0),\text{down}) + \gamma V((0,0)),$$
$$r((0,0),\text{left}) + \gamma V((0,0)),$$
$$r((0,0),\text{right}) + \gamma V((1,0))\}$$
$$= \text{argmax}_a\{0,0,0,0\}$$

We can randomly select one action from $\mathcal{A} = \{\text{up}, \text{down}, \text{left}, \text{right}\}$.
However, it is better select one action from up and right (why?).

# Policy Iteration

- Policy iteration improves the policy directly



A bad policy $\pi_2$

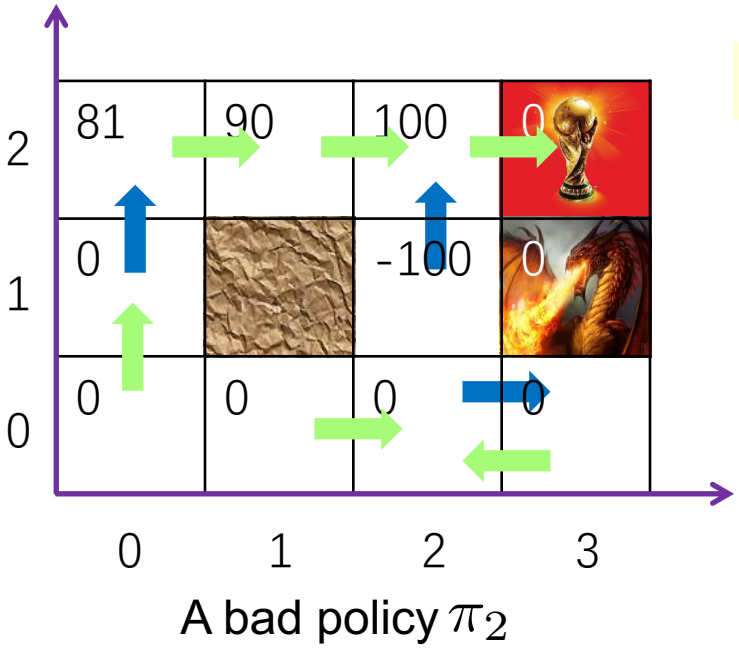
$1^{st}$ iteration: update the policy

$$\pi((0,0)) = \text{argmax}_a\{r((0,0),\text{up}) + \gamma V((0,1)),$$
$$r((0,0),\text{down}) + \gamma V((0,0)),$$
$$r((0,0),\text{left}) + \gamma V((0,0)),$$
$$r((0,0),\text{right}) + \gamma V((1,0))\}$$
$$= \text{argmax}_a\{0,0,0,0\}$$

We can indeed assign negative rewards for actions that will not alter the states when these states are not the goal states. Or, we can simply ignore these actions.

# Policy Iteration

- Policy iteration improves the policy directly




A bad policy $\pi_2$

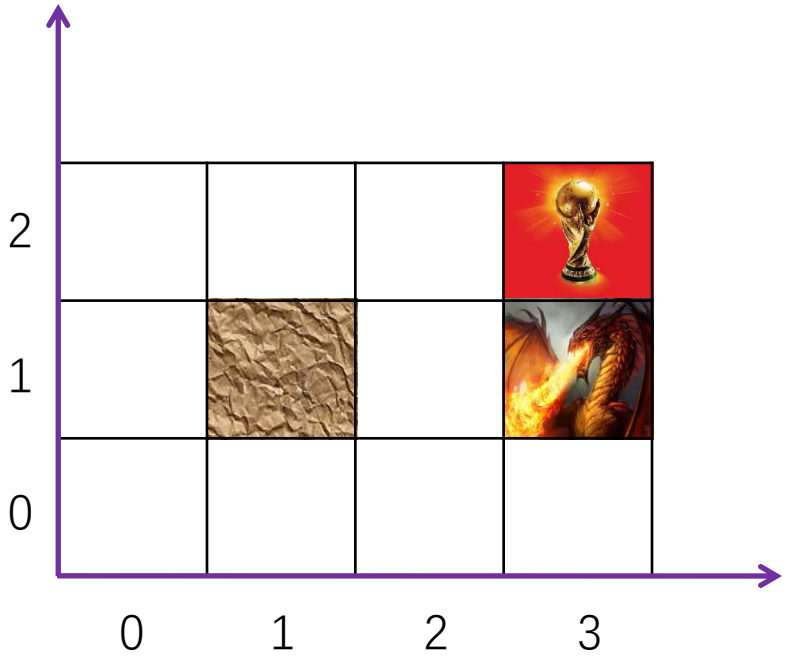
$1^{st}$ iteration: update the policy

$$\pi((0,0)) = \text{up}$$
$$\pi((1,0)) = \text{right}$$
$$\pi((2,0)) = \text{right}$$
$$\pi((3,0)) = \text{left}$$
$$\pi((0,1)) = \text{up}$$
$$\pi((2,1)) = \text{up}$$
$$\pi((3,1)) = \text{END}$$
$$\pi((0,2)) = \text{right}$$
$$\pi((1,2)) = \text{right}$$
$$\pi((2,2)) = \text{right}$$
$$\pi((3,2)) = \text{END}$$

# Learning Algorithms

# Learning

- Learning: as the environment model, i.e., the transition and reward, is unknown, the agent may need to learn them based on the training information.



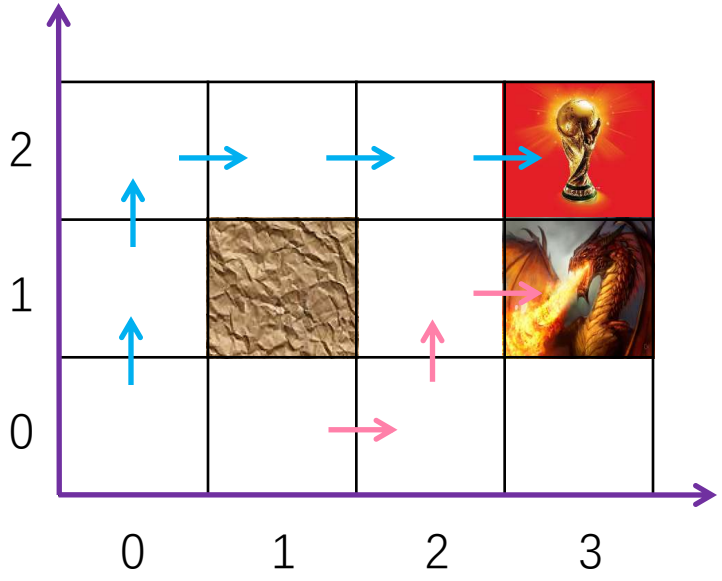

**Unknown**

$\delta : \mathcal{S} \times \mathcal{A} \to \mathcal{S}$ : state transition

$r : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ : reward

# Learning

- Learning: as the environment model, i.e., the transition and reward, is unknown, the agent may need to learn them based on the training information.
  - Model-free approach: the agent learns the optimal policy directly, e.g., Q-learning
  - Model-based approach: the agent first learns the environment model and then the optimal policy




Examples of training data

$$(0,0) \xrightarrow[0]{\text{up}} (0,1) \xrightarrow[0]{\text{up}} (0,2) \xrightarrow[0]{\text{right}} (1,2) \xrightarrow[0]{\text{right}} (2,3) \xrightarrow[100]{\text{right}} (3,2)$$

$$(1,0) \xrightarrow[0]{\text{right}} (2,0) \xrightarrow[0]{\text{up}} (2,1) \xrightarrow[-100]{\text{right}} (3,1)$$

# The Q-learning Algorithm

- Initialize the matrix $\hat{Q}$ to zero
- Observe the current state $s$
- Do forever:
  - Pick and perform an action $a$
  - Receive immediate reward $r$
  - Observe the new state $s'$
  - Update

$$\hat{Q}(s,a) \leftarrow r + \gamma \max_{a'} \hat{Q}(s',a')$$

  - $s \leftarrow s'$

A sufficient condition for $\hat{Q}(s,a)$ to converge is to visit each state-action pair infinitely often

# The Q-learning Algorithm

- Initialize the matrix $\hat{Q}$ to zero
- Observe the current state $s$
- Do forever:
  - Pick and perform an action $a$
  - Receive immediate reward $r$
  - Observe the new state $s'$
  - Update

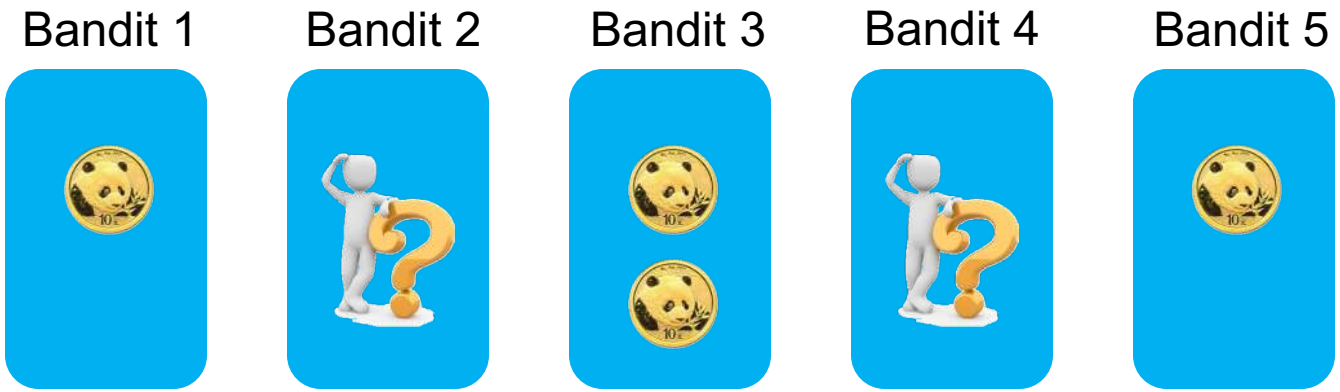
$$\hat{Q}(s,a) \leftarrow r + \gamma \max_{a'} \hat{Q}(s',a')$$

  - $s \leftarrow s'$

How to pick the action?

# Exploitation vs Exploration

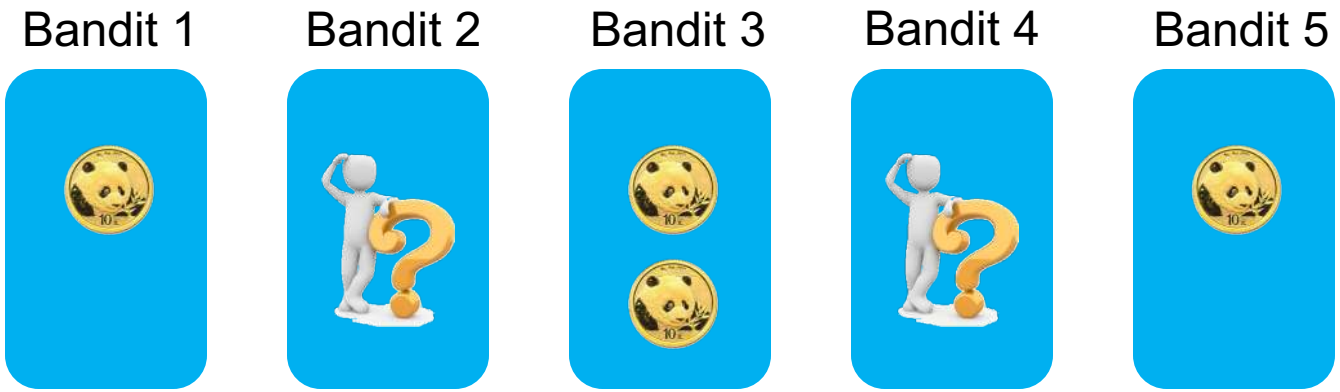
- Multi-armed bandit




- Which machine next?
  - Exploitation: the machine with the largest reward at present
  - Exploration: randomly select a machine

# Exploitation vs Exploration

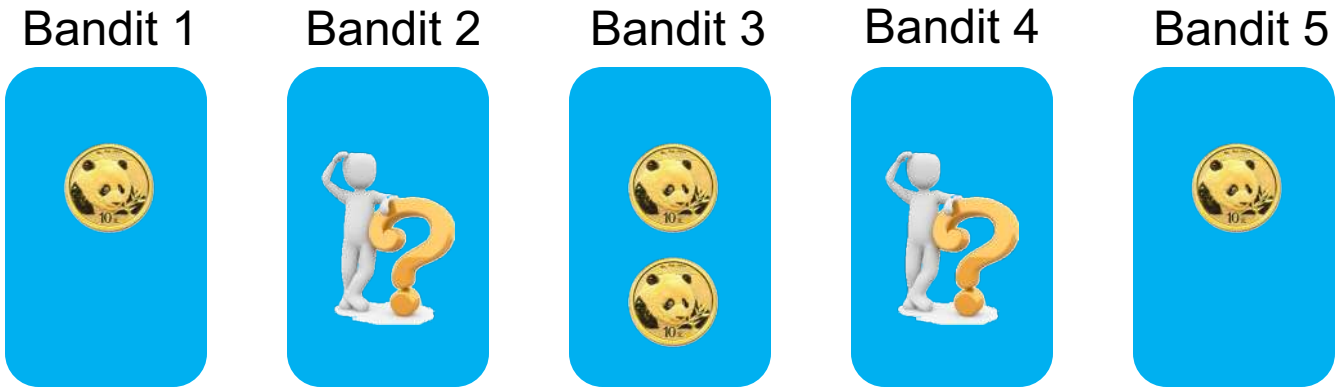
- Multi-armed bandit




- $\epsilon$-greedy
  - with probability $1 - \epsilon$, we do exploitation
  - with probability $\epsilon$, we do exploration, i.e., we uniformly randomly select an action from all possible actions
- Tips for $\epsilon$-greedy
  - At the beginning, the agent does not know the environment very well. Thus, it need to do more exploration and a large value of $\epsilon$ is needed.
  - When the environment model is well explored, the agent can do more exploitation. Thus, we favor a small value of $\epsilon$.

# Exploitation vs Exploration

- Multi-armed bandit

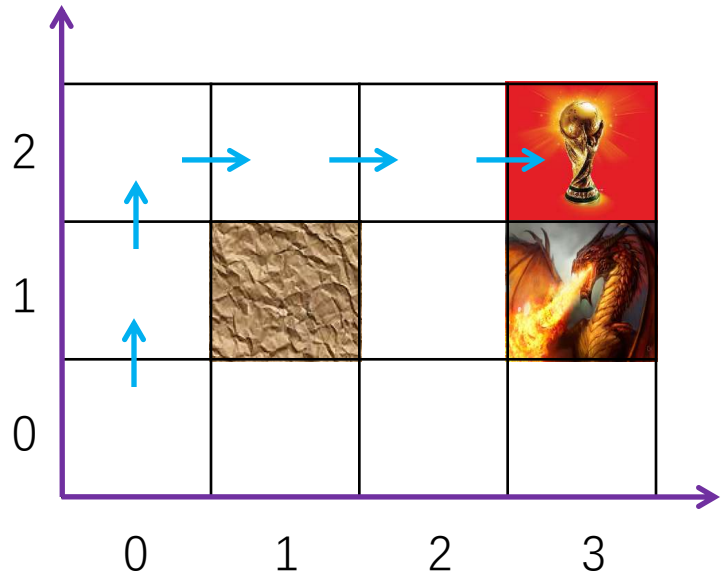



- A soft sampling strategy
  - Given a state, we can choose action probabilistically

$$\mathrm{P}[a|s] = \frac{e^{\hat{Q}(s,a)/T}}{\sum_{a'} e^{\hat{Q}(s,a')/T}}$$

  - Smaller values of $T$ will assign higher probabilities for actions with high $\hat{Q}$, leading to an exploitation strategy.
  - Larger values of $T$ will encourage the agent to explore actions that do not currently have high $\hat{Q}$ values.

# The Q-learning Algorithm

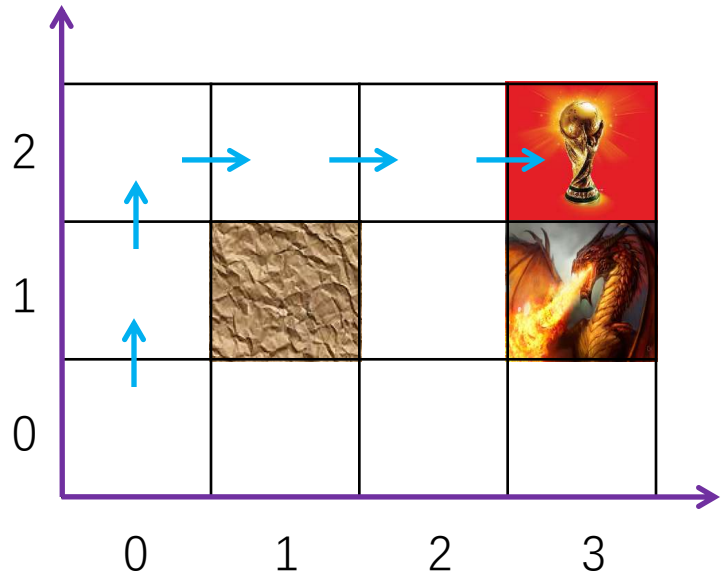


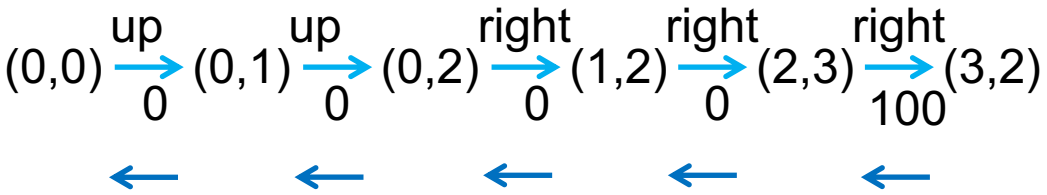

(0,0) —up, 0→ (0,1) —up, 0→ (0,2) —right, 0→ (1,2) —right, 0→ (2,3) —right, 100→ (3,2)

$\epsilon = 0.3$

- an example episode
- the initial state in each episode could NOT be fixed (why?)

# The Q-learning Algorithm



$\epsilon = 0.3$

# Questions