

## Problem

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Please select a problem to answer. We will evaluate your math skills according to your answer.

### Problem 1: Second-order sufficient optimality conditions

Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is twice continuously differentiable at  $\mathbf{x}$ . Show that  $\mathbf{x}$  is a strict local minimum if  $\nabla f(\mathbf{x}) = 0$  and the Hessian matrix  $\mathbf{H}(\mathbf{x})$  is positive definite.

### Problem 2: Low-rank approximation

Please solve the problem as follows

$$\min_{X \in \mathbb{R}^{m \times n}} \{\|A - X\|_F : \mathbf{rank}(X) \leq K\}.$$

### Problem 3: Random walk on $\mathbb{Z}$

Consider the random walk  $X = \{X_n\}_{n \geq 0}$  on  $\mathbb{Z}$  that starts at  $X_0 = 0$ . The particle moves with probability  $p$  one unit to the right and with probability  $q = 1 - p$  one unit to the left at each transition. Prove that the state 0 is recurrent (i.e.,  $\mathbb{P}\{X_n = 0 \text{ i.o.} | X_0 = 0\} = 1$ ) if and only if  $p = q = 1/2$ .